SEMI-SIMPLE RINGS AND COMMUTATIVITY (*)

by R.D. GIRI and V.G. BUJADE (in Nagpur)(**)

SOMMARIO. - *Sia R un anello semisemplice. Si prova che se per ogni coppia di elementi x, y in R esistono interi positivi $m = m(x,y)$ e $n = n(x,y)$ tali che $[[(yxy)^m, (xy)^n + (yx)^n], (yxy)] = 0$, allora R è commutativo.*

SUMMARY. - *Let R be a semi-simple ring. We prove that if for any pair of elements x, y in R, there exist positive integers $m = m(x,y)$ and $n = n(x,y)$ such that $[[(yxy)^m, (xy)^n + (yx)^n], (yxy)] = 0$, then R is commutative.*

1. Introduction.

In [4], Quadri and his team proved that in a semi-simple ring R if for given x, y in R there exist positive integers $m = m(x,y)$ and $n = n(x,y)$ such that $[x^m, (xy)^n] = [(yx)^n, x^m]$. Then R is commutative.

A much better generalisation of the above result can be obtained by considering the identity $[(yxy)^m, (xy)^n + (yx)^n] = 0$ and we have got it. But we can further weaken this identity by choosing its left hand side in the centre of R. In fact we prove below the best generalisation of all possible generalisations of Quadri [4].

THEOREM. *Let R be a semi-simple ring. Suppose that given x, y in R, there exist positive integers $m = m(x,y)$ and $n = n(x,y)$ such that $[[(yxy)^m, (xy)^n + (yx)^n], yxy] = 0$. Then R is commutative.*

Throughout this paper R is taken as an associative ring and $[a, b] = ab - ba$ for any pair a, b in R.

(*) Pervenuto in Redazione il 21 dicembre 1992.

(**) Indirizzo degli Autori: Department of Mathematics, Nagpur University Campus, Nagpur-440010 (M.S.), (India).

2. Preparatory Results.

In this section we develop those results which help us in proving the main theorem.

LEMMA 2.1. ([1]. Theorem 1). *Let R be a ring without nonzero nil right ideals. Suppose that given a, b in R, there exist positive integers $m = m(a, b) \geq 1$, $n = n(a, b) \geq 1$ and $t = t(a, b) \geq 1$ such that $[a^m, [a^n, b^t]] = 0$. Then R is commutative.*

LEMMA 2.2. *Let R be a division ring. Suppose that given x, y in R, there exist positive integers $m = m(x, y)$ and $n = n(x, y)$ such that $[[(yxy)^m, (xy)^n + (yx)^n], yxy] = 0$. Then R is commutative.*

Proof. Let $y \neq 0$, then by hypothesis there exist positive integers $m = m(y^{-1}xy^{-1}, y)$ and $n = n(y^{-1}xy^{-1}, y)$ such that the given identity reduces to

$$[[(yy^{-1}xy^{-1}y)^m, (y^{-1}xy^{-1}y)^n + (yy^{-1}xy^{-1})^n], yy^{-1}xy^{-1}y] = 0$$

i.e.

$$[[x^m, (y^{-1}x)^n + (xy^{-1})^n], x] = 0 \tag{1}$$

Replacing y by xy^{-1} in (1), we obtain $[[x^m, y^n + xy^nx^{-1}], x] = 0$, which on simplification yields

$$x^my^nx - y^nx^{m+1} - x^{m+2}y^nx^{-1} + x^2y^nx^{m-1} = 0 \tag{2}$$

Right multiplying (2) by x, we get

$$x^my^nx^2 - y^nx^{m+2} - x^{m+2}y^n + x^2y^nx^m = 0 \ .$$

Therefore $[x^2, [x^m, y^n]] = 0$. In a division ring lemma 2.1 is applicable, hence R is commutative.

3. Proof of the Theorem.

Suppose that R is a semi-simple ring such that for all x, y in R there exist positive integers $m = m(x, y)$ and $n = n(x, y)$ for which

$$[[(yxy)^m, (xy)^n + (yx)^n],\ (yxy)] = 0 \qquad \text{(A)}$$

A semi-simple ring is isomorphic to a subdirect sum of primitive rings. Further the identity (A) satisfied by a ring is also satisfied by all its subrings and homomorphic images. So to prove the theorem for semisimple rings it suffices to prove it for primitive rings. Now a primitive ring satisfying (A) is necessarily a division ring which can easily be checked. Because by Jacobson density theorem a primitive ring R is isomorphic to D_t where D is a division ring and $t > 1$ is an integer. But we find that identity (A) is not satisfied by this complete matrix ring on choosing $x = e_{11} + e_{12}$ and $y = e_{11}$ for $t = 2$. Thus our primitive ring is necessarily a division ring. Hence the theorem follows from lemma 2.2.

REFERENCES

[1] HONGAN M. and TOMINAGA H., *Some commutativity theorem for semiprime rings*, Hokkaido Math. J. **10** (1981), 271-277.

[2] JACOBSON N., Amer. Math. Society, Colloq. Publ. **37**, Providence, 1964.

[3] MCCOY N.H., *Theory of rings*, MacMillan Press, New York.

[4] QUADRI M.A. and KHAN M.A., *A theorem on commutativity of semi-simple rings*, Soochow Journal of Math. **11** (1985), 97-99.