STRUCTURE OF CERTAIN NEAR-RINGS (*)

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SOMMARIO. - *Recentemente Ligh e Luh* [6] *hanno trovato una scomposizione in somma diretta per gli anelli che hanno la proprietà* $(xy)^{n(x,y)} = xy$, *usando la commutatività di questi anelli provata da Searcóid e MacHale* [8]. *In questo lavoro si continua questo studio, e si ottiene una scomposizione per i quasi-anelli che hanno una qualunque delle proprietà (i)* $(xy)^{n(x,y)} = xy$ *(ii)* $x^{n(x,y)}y^{m(x,y)} = xy$ *e (iii)* $y^{m(x,y)}x^{n(x,y)} = xy$.

SUMMARY. - *Using commutativity of rings satisfying* $(xy)^{n(x,y)} = xy$, *proved by Searcóid and MacHale* [8], *recently Ligh and Luh* [6] *have given direct-sum decomposition for rings with the mentioned condition. More recently Bell and Ligh* [3] *sharpened this result and also established a structure of the near-rings satisfying* $(xy)^{n(x,y)} = yx$. *In the present paper we continue these investigations and obtain decomposition for near-rings satisfying any of the conditions (i)* $(xy)^{n(x,y)} = xy$ *(ii)* $x^{n(x,y)}y^{m(x,y)} = xy$ *and (iii)* $y^{m(x,y)}x^{n(x,y)} = xy$.

1. Introduction.

The property $x^n = x$ has been among the favourites of many ring theorists over the last few decades since Jacobson [5] first studied the commutativity of rings satisfying this condition in order to generalize the classical Wedderburn theorem [9]. The purpose of the present

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paper is to give decomposition theorems for near-rings satisfying any of the following conditions which are weaker than the above mentioned one (cf. [6]).

(P_1) For every pair of elements x, y in a ring R, there exist positive integers $m = m(x,y)$, $n = n(x,y)$ at least one of them greater than 1 such that $xy == y^m x^n$.

(P_2) For every pair of elements x, y in a ring R, there exists a positive integer $n = n(x,y) > 1$ such that $xy = (yx)^n$.

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2. Notations and Preliminaries.

Throughout the paper R is a left near-ring. An element x of R is said to be distributive if $(y+z)x = yx + zx$ for all y, z in R. If $xy = 0$ implies $yx = 0$, R is said to be zero-commutative and if for all $x \varepsilon R$, $0x = 0$, R is called zero-symmetric. (We may recall that the left distributivity of R yields $x0 = 0$).

A near-ring R is said to be periodic if for each x in R, there exist distinct positive integers n, m for which $x^n = x^m$.

An ideal of a near-ring R is a normal subgroup I of R^+ such that (i) $RI \subseteq I$ and (ii) $(x+i)y - xy \varepsilon I$ for all x, y in R and $i \varepsilon I$.

A near-ring R is an orthogonal-sum of sub-near rings A and B denoted by $R = A \dot{+} B$ if $AB = BA = \{0\}$ and each element of R has a unique representation in the form $a + b$, with $a \varepsilon A$ and $b \varepsilon B$.

Borrowing the notions of central and nilpotent elements from ring theory, we denote the multiplicative centre of R by Z and the set of nilpotent elements by N, The set $\{x \varepsilon R / x^{n(x)} = x$ for some positive integer $n(x) > 1\}$ of potent elements will be denoted by M.

3. Some Basic Results.

Following Lemmas are essentially proved in [1], [2] and [3] respectively.

LEMMA 1. Let R be a zero-symmetric near-ring satisfying the following properties:

i) For each x in R, there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$.

ii) Every non-trivial homomorphic image of R contains a non-zero central idempotent.

Then $(R, +)$ is commutative.

LEMMA 2. Let R be a zero-commutative near-ring. Then the set N of nilpotent elements is an ideal if and only if N is a subgroup of the additive group R^+.

LEMMA 3. Let R be a near-ring in which idempotents are multiplicatively central. If e and f are any idempotents, there exists an idempotent g such that $ge = e$ and $gf = f$.

Now we prove the following Lemma.

LEMMA 4. Let R be a near-ring in which for any $x, y \varepsilon R$, there exist positive integers $m = m(x, y)$ and $n = n(x, y)$ such that $xy = y^m x^n$. Then idempotents are central.

Proof. Let e be an idempotent and $x \varepsilon R$. Then by hypothesis there exist integers $p \geq 1$ and $q \geq 1$ such that $xe = e^p x^q$ and hence $xe = ex^q = exe$. similarly $ex = x^s e^r = x^s e$ for some integers $s \geq 1$ and $r \geq 1$. This implies that $ex = x^s e = exe$. Thus we find that $ex = xe$ and $e \varepsilon Z$.

4. Decomposition Theorems.

THEOREM 1. *Let R be a near-ring satisfying condition (P_1). Then N is a sub-near-ring with trivial multiplication, M is a sub-near-ring with $(M,+)$ abelian, and $R = M + N$.*

Proof. Notice that R satisfying (P_1) is necessarily zero-symmetric as well as zero-commutative. If $u \varepsilon N$ and $x \varepsilon R$, then, for some integers m_1 and n_1, $ux = x^{m_1}u^{n_1}$ where either $m_1 > 1$ or $n_1 > 1$. Now choose appropriate positive integers m_2 and n_2 at least one of them greater than 1 such that $x^{m_1}u^{n_1} = u^{n_1m_2}x^{m_1n_2}$ and hence $ux = u^{n_1m_2}x^{m_1n_2}$. It is now clear that for arbitrary t we have $ux = u^{n_1m_2\ldots n_{t-1}m_t}x^{m_1n_2\ldots m_{t-1}n_t}$ or $ux = x^{m_1n_2\ldots m_{t-1}n_t}u^{n_1m_2\ldots n_{t-1}m_t}$ according as t is even or odd, where either $m_1, m_2, \ldots, m_t > 1$ or $n_1, n_2, \ldots, n_t > 1$. Also from (P_1), we deduce that for all x in R, $x^2 = x^p$ for some integer $p \geq 2$. Hence, $u^2 = 0$ for all $u \varepsilon N$ and the above yields that $ux = 0$ for all x in R. Since R is zero-commutative, the nilpotent elements of R annihilate R on both sides i.e. $NR = RN = \{0\}$. Thus in particular $N^2 = \{0\}$, and if $u, v \varepsilon N$, then it follows that $(u-v)^2 = 0$. Now application of Lemma 2 yields that N is an ideal.

Let $r \varepsilon R$ such that $r^{n'+m'} = r^2$, where either $m' > 1$ or $n' > 1$. We can write $r = r - r^{n'+m'-1} + r^{n'+m'-1}$. Since $r(r - r^{n'+m'-1}) = 0$ and R is zero-commutative, we get $(r - r^{n'+m'-1})r = 0$ and $(r - r^{n'+m'-1})r^{n'+m'-1} = 0$. Hence $(r - r^{n'+m'-1})^2 = 0$ and $(r - r^{n'+m'-1}) \varepsilon N$.

Now

$$\begin{aligned}
(r^{n'+m'-1})^{n'+m'-1} &= r^{(n'+m'-1)(n'+m'-1)} \\
&= r^{(n'+m'-2)(n'+m')+1} \\
&= r^{(n'+m'-2)(n'+m')}r \\
&= (r^{n'+m'})^{(n'+m'-2)}r \\
&= (r^2)^{n'+m'-2}r
\end{aligned}$$

This yields,

$$(*) \qquad (r^{n'+m'-1})^{n'+m'-1} = (r^2)^{n'+m'-2}r$$

Now if $n' + m' - 2 > 1$, then $r^{n'+m'-2}$ is idempotent and $(r^{n'+m'-1})^{n'+m'-1} = r^{n'+m'-1}$ for $n' + m' - 1 > 1$, which gives that $r^{n'+m'-1} \varepsilon M$. Also if $n' + m' - 2 = 1$ so that $r^{n'+m'-1} = r^2$, then in view of (*) we have $r^3 = r^2$ and

$$(r^{n'+m'-1})^2 = (r^2)^2 = r^3 r = r^3 = r^2 = r^{n'+m'-1} .$$

Thus again $r^{n'+m'-1} \varepsilon M$. Hence in every case $R = M + N$.

Now we show that M is a sub-near-ring. Let $a, b \varepsilon M$ and choose integers $k = k(a) > 1$, $p = p(b) > 1$ such that $a^k = a$ and $b^p = b$. Then $e = a^{k-1}$ and $f = b^{p-1}$ are idempotents such that $ea = a$, $fb = b$. Using (P_1), we have $(ba)^2 = (ba)^q$ for some integer $q \geq 3$ that is $baba = ba(ba)^{q-2}ba$. Now, in view of Lemma 4, we get $ab = fabe = fa(ba)^{q-2}be = e(ab)^{q-1}f = (ab)^{q-1}$ where $q - 1 \geq 2$. Hence, $ab \varepsilon M$. Moreover, since R/N has $x^t = x$ property, we have an integer $j > 1$ such that

$$(a - b)^j = a - b + u, \ u \varepsilon N . \tag{1}$$

Using Lemma 3, choose an idempotent g for which $ge = e$, $gf = f$ and thus $ga = a$ and $gb = b$. Multiplying (1) by g we get $(a - b)^j = a - b$ i.e. $a - b \varepsilon M$, hence M is a sub-near-ring. Also by Lemma 1, $(M, +)$ is abelian. Trivially $M \cap N = \{0\}$. Let $a + u = b + v$, where $a, b \varepsilon M$ and $u, v \varepsilon N$. Then $a - b = v - u \varepsilon M \cap N = \{0\}$, which yields $a = b$ and $v = u$. Hence $R = M \dot{+} N$.

The result also holds if R satisfies the condition (P_2) which is Theorem 6 in [3].

The following example due to Clay [4, # 29(2.5)] shows that one cannot get a direct-sum decomposition under the hypotheses of the above theorem even in case of distributive near-rings.

EXAMPLE 1. Consider the non-abelian additive group $(R, +)$, isomorphic to the symmetric group S_3 and define the multiplication in R as follows:

$\cdot$	0	a_1	a_2	a_3	a_4	a_5
0	0	0	0	0	0	0
a_1	0	a_1	a_1	a_1	0	0
a_2	0	a_1	a_1	a_1	0	0
a_3	0	a_1	a_1	a_1	0	0
a_4	0	0	0	0	0	0
a_5	0	0	0	0	0	0

Then $\langle R,+,\cdot\rangle$ is a commutative near-ring satisfying $(ba)^2 = b^2a^2 = ab$ for all a, b in R. However, $M = \{0, a_1\}$ is not an ideal of R.

Further, if R satisfies either of the conditions (P_3) and (P_4), then we may not get even the orthogonal-sum decomposition of R which is evident from the following example.

EXAMPLE 2. Let $R = \{0, a, b, c\}$ with addition and multiplication tables defined as follows:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	0
c	0	c	0	c

It is easy to notice that R is a near-ring satisfying both the conditions (P_3) and (P_4). But the set $M = \{0, a, c\}$ is not a sub-near-ring of R.

However, the following results can be proved proceeding on the same lines as above which show that under some extra hypotheses either of the conditions (P_3) and (P_4) also guarantees orthogonal-sum decomposition of the near-rings.

THEOREM 2. Let R be a zero-symmetric near-ring satisfying condition (P_3). If idempotent elements of R are central, then N is a sub-near-ring with trivial multiplication, M is a sub-near ring with $(M,+)$ abelian and $R = M \dot{+} N$.

THEOREM 3. Let R be a zero-commutative near-ring satisfying condition (P_4). If idempotent elements of R are central, then N is a sub-near-ring with trivial multiplication, M is a sub-near-ring with $(M,+)$ abelian and $R = M \dot{+} N$.

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