RIESZ TRANSFORMS AND MAXIMAL FUNCTIONS IN WEIGHTED ZYGMUND SPACES (*)

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SOMMARIO. - *Questo lavoro da una descrizione completa di una classe di funzioni peso w per le quali le trasformate di Riesz mandano $L \log^+ L(w)$ in $L^1(w)$ o L^1 debole.*

SUMMARY. - *The paper gives a complete description of a class of weight functions w for which Riesz transforms map $L \log^+ L(w)$ into $L^1(w)$ or weak L^1.*

Riesz transforms $R_j f$, $j = 1, 2, \ldots, n$, for a measurable function $f : \mathbb{R}^n \to \mathbb{R}^1$ with the condition

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^n} dx < \infty$$

are determined as follows:

$$R_j f(x) = \lim_{\varepsilon \to 0} c_n \int_{|y|>\varepsilon} \frac{y_j}{|y|^{n+1}} f(x-y) dy \ .$$

where $y = (y_1 \ldots, y_n)$ and $c_n = \pi^{-(n+1)/2}\Gamma(\frac{n+1}{2})$.

For a locally integrable function $f : \mathbb{R}^n \to \mathbb{R}^1$ the maximal function is determined by the equality

$$Mf(x) = \sup \frac{1}{|Q|} \int_Q |f(y)| dy \ ,$$

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where the exact upper bound is taken over all n-dimensional cubes Q with faces parallel to the coordinate axes and containing the point x.

We have

THEOREM 1. *Let* $w : \mathbb{R}^n \to \mathbb{R}^1$ *be a positive locally integrable function. The conditions below are equivalent:*

(i) *there exists a constant* $c_1 > 0$ *such that for any measurable function* f *supported in a cube* Q *we have the inequality*

$$\int_Q |R_j f(x)| w(x) dx \le c_1 \left\{ \int_Q w(x) dx + \int_Q |f(x)| \log^+ |f(x)| w(x) dx \right\},$$

where c_1 *is independent of* Q *and* f;

(ii) *there exist constants* $\varepsilon > 0$ *and* $c_2 > 0$ *such that for any cube* Q *and any measurable subset* $E \subseteq Q$ *we have*

$$\frac{1}{|Q|} \int_Q \exp\left(\varepsilon \frac{|R_j(\chi_E w)(x)|}{w(x)} \right) dx \le c_2, \; j = 1, 2, \dots, n .$$

THEOREM 2. *Let* $w : \mathbb{R}^n \to \mathbb{R}^1$ *be a positive locally integrable function and* $0 < \alpha < \infty$. *Then the following conditions are equivalent:*

(i) *there exists a constant* $c_3 > 0$ *such that for any* $\lambda > 0$ *and any measurable function* f *we have the estimate*

$$w\{x : |R_j f(x)| > \lambda\} \le$$

$$c_3 \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \frac{|f(x)|}{\lambda} \right)^\alpha w(x) dx, \quad j = 1, \dots, n ;$$

(ii) *there exists a constant* $c_4 > 0$ *such that for any* $\lambda > 0$ *and any measurable function* f *we have the estimate*

$$w\{x : |Mf(x)| > \lambda\} \le c_4 \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \frac{|f(x)|}{\lambda} \right)^\alpha w(x) dx ;$$

(iii) *there exists a constant* $\varepsilon > 0$ *such that*

$$\sup \frac{1}{|Q|} \int_Q \exp\left\{ \varepsilon \frac{1}{|Q| w(x)} \int_Q w(y) dy \right\}^{1/\alpha} dx < \infty ,$$

where the supremum is taken over all cubes Q.

Similar results for maximal functions were previously discussed in [1].

Let $\varphi : [0,\infty) \to [0,\infty)$ be a positive increasing function such that $\varphi(2t) \le c\varphi(t)$ for all $t > 0$. Let $1 \le p \le \infty$. We denote by $L^{p,\varphi}$ the space of locally integrable functions $f : \mathbb{R}^n \to \mathbb{R}^1$ such that

$$\|f\|_{L^{p,\varphi}} = \sup \left(\frac{1}{\varphi(|Q|)} \int_Q |f(x) - f_Q|^p dx \right)^{1/p} < \infty \,,$$

where $f_Q = \frac{1}{|Q|} \int f(x)dx$ and supremum is taken over all cubes Q. We have:

THEOREM 3. *Let $1 < p < \infty$ and the function $\varphi(t)t^{-1-p/n}$ be non-increasing. If $f \in L^{p,\varphi}$ and if Mf is not infinite everywhere, then we have that*

$$\|Mf\|_{L^{p,\varphi}} \le c\|f\|_{L^{p,\varphi}} \,,$$

where c is independent of f.

The function $\varphi(t) = t^\lambda$ satisfies the conditions of Theorem 3 for $0 \le \lambda \le 1 + p/n$. Thus, the above Theorem can be applied to $L^{p,\lambda}$ spaces for $0 \le \lambda \le 1 + p/n$, and we obtain a new proof of results given in [2,3].

Similar results for convolution operators of certain type were discussed previously in [4].

For a locally integrable function $f : \mathbb{R}^n \to \mathbb{R}^1$ the sharp function can be defined by the equality

$$f^{\#}(x) = \sup \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy \,,$$

where the supremum is taken over all cubes Q containing x.

Let $\varphi : [0,\infty) \to [0,\infty)$ be a positive increasing function, and $\varphi(0) = 0$. φ is called quasi convex if there exists a convex function ψ and a constant $c > 1$, such that, for all $t \in [0, +\infty)$

$$\psi(t) \le \varphi(t) \le c\psi(ct) \,.$$

We have:

THEOREM 4. *Let $\varphi : [0,\infty) \to [0,\infty)$ be an increasing function, and $\varphi(0) = 0$. For the existence of a constant $c > 0$ such that the inequality*

$$\int_{\mathbb{R}^n} \left(\varphi \circ (Mf)^{\#}\right)(x)dx \leq c \int_{\mathbb{R}^n} \left(\varphi \circ cf^{\#}\right)(x)dx$$

is valid for all measurable f, it is necessary and sufficient that the function φ^{α} be quasi convex for some $\alpha \in (0,1)$.

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