ON CERTAIN BILATERAL GENERATING FUNCTIONS (*)

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SOMMARIO . - *In questo lavoro si prova un teorema sulle funzioni bilaterali generanti per polinomi estesi di Jacobi, dal punto di vista dei gruppi di Lie. Vengono inoltre discussi alcuni casi speciali e un'applicazione del risultato.*

SUMMARY. - *In this note we prove a theorem on bilateral generating functions of extended Jacobi polynomials from Lie group view point. Some special cases and an application of our result are also discussed.*

1. Introduction.

The extended Jacobi polynomial as defined in [3] is

$$F_n(\alpha,\beta;x) = \frac{(-1)^n}{n!}\left(\frac{\lambda}{b-a}\right)^n (x-a)^{-\alpha}(b-x)^{-\beta} \times D^n\left[(x-a)^{\alpha+n}(b-x)^{\beta+n}\right],\ D=\frac{d}{dx}\,. \tag{1.1}$$

The aim at writing this article is to derive an unified presentation of bilateral generating functions for some special functions in terms of $F_n(\alpha-n;\beta;x)$ – the modified form of extended Jacobi polynomials which does not seem to have appeared in the earlier works.

The main result of this article is stated in the following theorem. In fact, it is mentioned that our result with the help of limiting process yields the corresponding bilateral generating functions involving Laguerre, Hermite, Bessel and Jacobi polynomials.

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THEOREM. *If*

$$G(x,w)=\sum_{n=0}^{\infty} a_n F_n(\alpha-n,\beta;x)w^n \tag{1.2}$$

then

$$(1-\lambda t)^{\alpha}\left(1-\lambda\frac{x-a}{b-a}t\right)^{-\alpha-\beta-1} G\left(\frac{x-\lambda bt\frac{x-a}{b-a}}{1-\lambda t\frac{x-a}{b-a}}, \frac{zt}{1-t\lambda}\right) \tag{1.3}$$

$$=\sum_{n=0}^{\infty} F_n(\alpha-n,\beta;x)\sigma_n(z)t^n$$

where

$$\sigma_n(z)=\sum_{k=0}^{n} a_k\binom{n}{k} z^k .$$

The importance of the above theorem lies in the fact that whenever one knows a generating relation of the form (1.2), the corresponding bilateral generating relation can at once be written down from (1.3). Thus one can get a large number of generating relations by attributing different suitable values to a_n in (1.2).

2. Proof of the Theorem.

From [2], we have

$$e^{wR}f(x,y) = (1+\lambda wy)^{\alpha}\left\{1+w\lambda\frac{x-a}{b-a}y\right\}^{-\alpha-\beta-1} \times f\left(\frac{x+\lambda wb\frac{x-a}{b-a}y}{1+\lambda w\frac{x-a}{b-a}y}, \frac{y}{1+w\lambda y}\right) \tag{2.1}$$

where

$$R=\frac{\lambda}{b-a}(x-a)(b-x)y\frac{\partial}{\partial x}-\lambda y^2\frac{\partial}{\partial y}+[(b-x)\alpha-(\beta+1)(x-a)]\frac{\lambda y}{b-a}$$

such that

$$R(F_n(\alpha-n;\beta;x)y^n) = -(n+1)F_{n+1}(\alpha-n-1,\beta;x)y^{n+1} \,. \tag{2.2}$$

Consider

$$G(x,w) = \sum_{n=0}^{\infty} a_n F_n(\alpha-n,\beta;x)w^n \,. \tag{2.3}$$

Now replacing w by wyz and then operating $(\exp(wR))$ on both sides of (2.3) we get

$$e^{wR}G(x,wyz) = e^{wR}\sum_{n=0}^{\infty} a_n(wz)^n F_n(\alpha-n,\beta;x)y^n \,. \tag{2.4}$$

The left member of (2.4) is

$$(1+\lambda wy)^{\alpha}\left(1+\lambda wy\frac{x-a}{b-a}\right)^{-\alpha-\beta-1} \tag{2.5}$$

$$\times G\left(\frac{x+\lambda bwy\frac{x-a}{b-a}}{1+\lambda wy\frac{x-a}{b-a}},\ \frac{wyz}{1+\lambda wy}\right) \,.$$

The right member of (2.4) is

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n(wz)^n\frac{w^k}{k!}R^k(F_n(\alpha-n,\beta;x)y^n) \tag{2.6}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n(wz)^n\frac{w^k}{k!}(-1)^k(n+1)_k F_{n+k}(\alpha-n-k;\beta;x)y^{n+k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n z^n\frac{(wy)^{n+k}}{k!}(-1)^k(n+1)_k F_{n+k}(\alpha-n-k,\beta;x)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n(-z)^n\frac{(wy)^{n+k}}{k!}(n+1)_k F_{n+k}(\alpha-n-k;\beta;x)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{n-k}(-z)^{n-k}\frac{(-wy)^n}{k!}(n-k+1)_k F_n(\alpha-n,\beta;x)$$

$$= \sum_{n=0}^{\infty}(-wy)^n F_n(\alpha-n,\beta;x)\sum_{k=0}^{n} a_{n-k}\frac{(n-k+1)_k}{k!}(-z)^{n-k} \,.$$

Equating the two members (2.5) and (2.6), we get

$$\begin{aligned}(1+\lambda wy)^{\alpha}&\left(1+\lambda wy\frac{x-a}{b-a}\right)^{-\alpha-\beta-1} \qquad (2.7)\\ &\times G\left(\frac{x+\lambda bwy\frac{x-a}{b-a}}{1+\lambda wy\frac{x-a}{b-a}}, \quad \frac{wyz}{1+\lambda wy}\right)\\ =&\sum_{n=0}^{\infty}(-wy)^n F_n(\alpha-n,\beta;x)\sum_{k=0}^{n}a_{n-k}\frac{(n-k+1)_k}{k!}(-z)^{n-k},\end{aligned}$$

replacing wy by "$-t$" and $(-z)$ by z, we get

$$(1-\lambda t)^{\alpha}\left(1-\lambda t\frac{x-a}{b-a}\right)^{-\alpha-\beta-1} G\left(\frac{x-\lambda bt\frac{x-a}{b-a}}{1-\lambda t\frac{x-a}{b-a}}, \quad \frac{zt}{1-\lambda t}\right) \qquad (2.8)$$

$$=\sum_{n=0}^{\infty}t^n F_n(\alpha-n,\beta;x)\sigma_n(z)$$

where

$$\sigma_n(z)=\sum_{k=0}^{n}a_k\binom{n}{k}z^k,$$

this completes the proof of the theorem.

3. Application.

As an application we consider the following generating relation [4]:

$$(1-\lambda t)^{\alpha}\left(1-\lambda t\frac{x-a}{b-a}\right)^{-\alpha-\beta-1}=\sum_{n=0}^{\infty}F_n(\alpha-n,\beta;x)t^n. \qquad (3.1)$$

If in our theorem, we take $a_n=1$, then

$$G(x,w)=(1-\lambda w)^{\alpha}\left(1-\lambda w\frac{x-a}{b-a}\right)^{-\alpha-\beta-1}.$$

So by the application of our theorem, we get the following generalization:

$$\{1-\lambda t(1+z)\}^{\alpha}\left\{1-\frac{\lambda t}{b-a}(x-a)(1+z)\right\}^{-\alpha-\beta-1} \qquad (3.2)$$

$$= \sum_{n=0}^{\infty} F_n(\alpha - n, \beta; x)\sigma_n(z)t^n$$

where

$$\sigma_n(z) = \sum_{n=0}^{n} a_k \binom{n}{k} z^k .$$

4. Some Special Cases.

SPECIAL CASE 1: Putting $-a = b = 1$, $\lambda = 1$, we get the following result on bilateral generating functions involving Jacobi polynomials:

RESULT 1: If

$$G(x, w) = \sum_{n=0}^{\infty} a_n P_n^{(\beta, \alpha - n)}(x) w^n \tag{4.1}$$

then

$$(1-t)^{\alpha} \left\{1 - \frac{t}{2}(x+1)\right\}^{-\alpha-\beta-1} G\left(\frac{x - \frac{t}{2}(x+1)}{1 - \frac{t}{2}(x+1)}, \frac{zt}{1-t}\right) \tag{4.2}$$

$$= \sum_{n=0}^{\infty} P_n^{(\beta, \alpha - n)}(x)\sigma_n(z)t^n$$

where

$$\sigma_n(z) = \sum_{k=0}^{\infty} a_k \binom{n}{k} z^k ,$$

which is found derived in [1].

SPECIAL CASE 2: Putting $a = 0$, $\beta = b$ and $\lambda = 1$ and then taking limit as $b \to \infty$ we get the following result on bilateral generating function involving Laguerre polynomial:

RESULT 2: If

$$G(x, w) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha - n)}(x) w^n \tag{4.3}$$

then

$$(1+t)^{\alpha}\exp(-xt)G\left(x(1+t),\frac{zt}{1+t}\right) \tag{4.4}$$
$$= \sum_{k=0}^{\infty} L_n^{(\alpha-n)}(x)\sigma_n(z)t^n$$

where

$$\sigma_n(z)=\sum_{n=0}^{\infty} a_k \binom{n}{k} z^k ,$$

which is found derived in [2].

SPECIAL CASE 3: Putting $\alpha = \beta$, $b = -a = \sqrt{\alpha}$ and recalling $\lambda = \frac{2}{\sqrt{a}}$ and then taking limit as $\alpha \to \infty$ we get the following result on bilateral generating function involving Hermite polynomial:

RESULT 3: If

$$G(x,w)=\sum_{n=0}^{\infty} a_n \frac{H_n(x)}{n!} w^n \tag{4.5}$$

then

$$\exp(2xt-t^2)G(x-t,zt)=\sum_{n=0}^{\infty} H_n(x)\sigma_n(z)\frac{t^n}{n!} \tag{4.6}$$

where

$$\sigma_n(z)=\sum_{n=0}^{\infty} a_k \binom{n}{k} z^k ,$$

which is found derived in [5].

SPECIAL CASE 4: Putting $b = -a = 1$, $\lambda = 1$, $\alpha = \nu - \varepsilon - 1$, $\beta = \varepsilon - 1$ and replacing x by $(1 + 2x\varepsilon/s)$ and t by sw/ε and then taking limit as $\varepsilon \to \infty$ in our theorem and finally using the relation [3]:

$$Lt_{\varepsilon\to\infty}\frac{\overline{|n+1}}{\varepsilon^n}F_n\left(\nu\ -\varepsilon -\ 1,\ \varepsilon -\ 1;\ 1+\frac{2x\varepsilon}{s}\right)=Y_n(x,\nu,s) \tag{4.7}$$

we get the following result on bilateral generating function involving Bessel polynomial: If

$$G(x,w) = \sum_{n=0}^{\infty} a_n Y_n(x, \alpha - n; s)\frac{(sw)^n}{n!} \tag{4.8}$$

then

$$\begin{aligned} &\exp(sw)(1-xw)^{-\nu+1} G\left(\frac{x}{1-xw}, wz\right) \\ &= \sum_{n=0}^{\infty} Y_n(x, \alpha - n; s)\frac{(ws)^n}{n!}\sigma_n(z) \end{aligned} \tag{4.9}$$

where

$$\sigma_n(z) = \sum_{n=0}^{\infty} a_k \binom{n}{k} z^k .$$

The above result is noteworthy.

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