

# THE TENSOR PRODUCT OF HOPF ALGEBRAS(\*)

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**SOMMARIO.** - *Si introducono le costruzioni della "hom" interna e il prodotto tensoriale di algebre di Hopf abeliane. Queste sono analoghe a quelle dei gruppi abeliani, e dimostrano che le algebre di Hopf abeliane sono una categoria monoidale e chiusa. Si forniscono esempi di schemi di gruppi affini.*

**SUMMARY.** - *We here construct the internal "hom" and tensor product of abelian Hopf algebras over a field. These are analogous to those for abelian groups and show that abelian Hopf algebras form a monoidal closed category. Affine group schemes are given as examples.*

The category  $\mathcal{H}$  of commutative and cocommutative Hopf algebras over a field is well known to be abelian. It is the category of abelian group objects in the category of cocommutative coalgebras. Hence it is natural to construct the tensor product of two abelian Hopf algebras. This is not to say the tensor product over the base field, which yields the biproduct in  $\mathcal{H}$ , but the bifunctor which classifies coalgebra maps that are multiplicative in each variable. Since the group structure on a Hopf algebra comes from its multiplication, this is the natural analogue of the tensor product of abelian groups. We will show that this tensor, denoted  $\otimes$ , is left adjoint to another natural functor, the internal Hom on  $\mathcal{H}$ .

These constructions are well known in the setting of monoidal closed categories [2], and the tensor  $\otimes$  appears implicitly in the study of the Hopf rings of complex cobordism [9,12]. We here establish the

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basic properties of the tensor and internal Hom and give examples in the realm of affine group schemes.

Let  $k$  be a field, and let  $\mathcal{V}$  denote the category of  $k$ -vector spaces. Let  $\mathcal{C}$  denote the category of cocommutative, coassociative, counitary  $k$ -coalgebras. The diagonal and counit maps of a coalgebra will be denoted  $\Delta$  and  $\varepsilon$  respectively. An abelian Hopf algebra  $H$  comes equipped with an associative and commutative product  $m : H \otimes H \rightarrow H$ , a unit  $1 : k \rightarrow H$ , and inverses, i.e. an involution  $S : H \rightarrow H$  satisfying  $\Delta(I \otimes S)m = \varepsilon 1$ . This  $S$  is the inverse of the identity map  $I : H \rightarrow H$  in the group  $\text{Hom}_{\mathcal{C}}(H, H)$ . When working with algebraic categories over  $\mathcal{V}$ , one should think of  $\mathcal{C}$  as replacing  $\mathcal{S}$ , the category of sets. Each category of algebras has a  $\mathcal{C}$ -valued Hom functor [3],  $\mathcal{C}$  is cartesian closed [13] and may be used to parametrize families of objects arising in algebra [7]. The operations on the objects of  $\mathcal{C}$  reflect our concern with linear phenomena. Likewise, one should think of  $\mathcal{H}$  as replacing the category of abelian groups.

## 1. Tensor and Hom.

We shall first construct the internal Hom in  $\mathcal{H}$  which is the  $\mathcal{C}$ -valued Hom with its natural Hopf structure. Given Hopf algebras  $G$  and  $H$ , let  $\mathcal{V}(G, H)$  denote the cofree coalgebra over  $\text{Hom}_{\mathcal{V}}(G, H)$ . The adjunction  $\mathcal{V}(G, H) \rightarrow \text{Hom}_{\mathcal{V}}(G, H)$  yields an evaluation map  $G \otimes \mathcal{V}(G, H) \rightarrow H$ , and  $\mathcal{V}(-, -)$  is a functor in the obvious way; it defines the enrichment of  $\mathcal{V}$  over  $\mathcal{C}$ . Note that each linear map  $\varphi : k \rightarrow \text{Hom}_{\mathcal{V}}(G, H)$  induces a unique coalgebra map  $\bar{\varphi} : k \rightarrow \mathcal{V}(G, H)$ . Now  $\mathcal{H}(G, H)$  will be a subobject of  $\mathcal{V}(G, H)$ , namely the joint equalizer in  $\mathcal{C}$  of the following five pairs of maps:  $\{(I, \Delta), \Delta \otimes (\Delta, I)\}$ ,  $\{(I, \varepsilon), (\varepsilon, \bar{\varepsilon}\varepsilon)\}$ ,  $\{(m, I), \Delta \otimes (I, m)\}$ ,  $\{(1, I), \varepsilon \bar{1}\}$ ,  $\{(S, I), (I, S)\}$ . This definition says that  $\mathcal{H}(G, H)$  contains the elements of  $\mathcal{V}(G, H)$  that respect Hopf algebra structure maps.

The coalgebra maps  $k \rightarrow \mathcal{H}(G, H)$  are determined by the elements  $f$  of  $\mathcal{H}(G, H)$  such that  $f\Delta = f \otimes f$  and  $f\varepsilon = 1$ , i.e. the *points* of  $\mathcal{H}(G, H)$ . From the definition of  $\mathcal{H}(G, H)$  as an equalizer we see that its points are in one-to-one correspondence with the Hopf algebra maps  $G \rightarrow H$ . Thus, the  $\mathcal{S}$ -valued functor  $\text{Hom}_{\mathcal{H}}(-, -)$  may be recovered as  $\mathcal{H}(-, -)$  followed by  $\text{Hom}_{\mathcal{C}}(k, -) : \mathcal{C} \rightarrow \mathcal{S}$ . Further-

more, internal convolution, i.e. the map

$$\otimes \cdot (\Delta, m) : \mathcal{H}(G, H) \otimes \mathcal{H}(G, H) \rightarrow \mathcal{H}(G \otimes G, H \otimes H) \rightarrow \mathcal{H}(G, H)$$

and internal composition with the antipode of  $H$  give  $\mathcal{H}(G, H)$  the structure of a Hopf algebra.  $\mathcal{H}(G, H)$  contains all representative corepresentative functions from  $G$  to  $H$ .

**THEOREM 1.1.**  *$\mathcal{H}$  is a closed category, with its internal Hom functor given by  $\mathcal{H}(-, -)$ . For each Hopf algebra  $G$ ,  $\mathcal{H}(G, -) : \mathcal{H} \rightarrow \mathcal{H}$  has a left adjoint,  $- \otimes G$ , the tensor product in  $\mathcal{H}$ .*

*Proof.* The composition comes from that in  $\mathcal{C}$  [3]. Since each step in the construction of  $\mathcal{H}(G, -)$  preserves limits, and limits in  $\mathcal{H}$  are computed in  $\mathcal{C}$ ,  $\mathcal{H}(G, -)$  has a left adjoint by the adjoint functor theorem.  $\diamond$

By mimicking the construction of the tensor product of abelian groups we may give a more enlightening description of  $\otimes$ . The free Hopf algebra functor  $F : \mathcal{C} \rightarrow \mathcal{H}$  gives the left adjoint for the forgetful functor [11].  $G \otimes H$  is the quotient of  $(G \otimes H)F$  which classifies coalgebra maps from  $G \otimes H$  that are Hopf algebra maps in each variable. In terms of generators and relations,  $G \otimes H$  may be described as the symmetric algebra generated by symbols  $g \otimes h$  modulo the usual relations for the tensor product over  $k$  as well as

$$g \otimes hh' = \sum (g_1 \otimes h)(g_2 \otimes h') \quad gg' \otimes h = \sum (g \otimes h_1)(g' \otimes h_2)$$

where  $g\Delta = \sum g_1 \otimes g_2$  and  $h\Delta = \sum h_1 \otimes h_2$ . The Hopf algebra structure on  $G \otimes H$  is given by the following equations, the last of which follows from lemma 1.2 below:

$$\begin{aligned} (g \otimes h)\Delta &= \sum (g_1 \otimes h_1) \otimes (g_2 \otimes h_2) \\ (g \otimes h)\varepsilon &= g\varepsilon h\varepsilon \\ 1_{G \otimes H} &= 1 \otimes 1 \\ (g \otimes h)S &= (gS \otimes h) = (g \otimes hS) \end{aligned}$$

**LEMMA 1.2.** *If  $g$  and  $h$  are elements of  $G$  and  $H$  respectively, then  $g \otimes 1 = g\varepsilon(1 \otimes 1)$  and  $1 \otimes h = h\varepsilon(1 \otimes 1)$  in  $G \otimes H$ .*

*Proof.*  $g\varepsilon(1\otimes 1) = (g\otimes 1)\varepsilon 1 = \Sigma(g_1\otimes 1)S(g_2\otimes 1) = \Sigma(g_1\otimes 1)S(g_2\otimes 1)(g_3\otimes 1) = \Sigma(g_1\otimes 1)\varepsilon 1(g_2\otimes 1) = g\otimes 1$  using the basic properties of the unit and the antipode  $S$ . The tensor is clearly symmetric.  $\diamond$

Recall that  $h$  in  $H$  is a *primitive* if  $h\Delta = h \otimes 1 + 1 \otimes h$  and  $h\varepsilon = 0$ . An  $n$ -sequence of divided powers in  $H$  is a sequence  $\{{}^m h\}_{m \leq n}$  of elements such that  ${}^m h\Delta = \sum_{i=0}^m {}^i h \otimes {}^{m-i} h$ , where  ${}^0 h = 1$  and  ${}^m h\varepsilon = 0$  for all  $1 \leq m \leq n$  [10]. Since sequences of divided powers are the key to the structure of any Hopf algebra (in characteristic zero, they generate the whole Hopf algebra), it behooves us to identify those of  $G\otimes H$ . If  $g$  and  $h$  are primitives, then so is  $g\otimes h$  though not all primitives of  $G\otimes H$  need be of this form, while the bilinearity of  $\otimes$  yields  $(g\otimes h^2) = (1\otimes h)(g\otimes h) + (g\otimes h)(1\otimes h) = 0$  by lemma 2.

**THEOREM 1.3.** (The calculus of points and primitives in  $G\otimes H$ ).  
Suppose  $x$  and  $y$  are points,  $g$  and  $h$  are primitives, and  $\{{}^m g\}$  and  $\{{}^n h\}$  are sequences of divided powers in  $G$  and  $H$  respectively. Then in  $G\otimes H$  we have:

1.  $x\otimes y$  is a point and  $x^m\otimes y^n = (x\otimes y)^{mn}$ .
2.  $x\otimes h$  is a primitive,  $x^m\otimes h = m(x\otimes h)$  and  $x\otimes h^m = (x\otimes h)^m$ .
3.  $g\otimes h$  is a primitive,  $g^m\otimes h^n = 0$  if  $m \neq n$ , and  $g^m\otimes h^m = m!(g\otimes h)^m$ .
4.  $\{{}^m g\otimes {}^n h\}$  is a sequence of divided powers.

*Proof.* Use lemma 1.2, the defining relations given above, and induction.  $\diamond$

## 2. Examples.

Using the relations given above it is easy to calculate the tensor products of the elementary commutative affine group schemes.  $G_m$  is the unit object for  $\otimes$ , i.e.  $G_m\otimes H \cong H$  for all  $H$ . We leave it to the reader to verify that  $G_a\otimes G_a \cong G_a$ ,  $\mu_p\otimes a_p \cong a_p$ ,  $A_k^m\otimes A_k^n \cong A_k^{mn}$ ,

and quite surprisingly  $a_p \circledast a_p \cong G_a$ . If  $W_1 = k[t, s]$  is the first Witt Hopf algebra [8] and  $h$  is a primitive in  $H$ , then  $h^p \circledast s = (h \circledast t)^p$  in  $H \circledast W_1$ . The case of two schemes of multiplicative type is taken care of by the following easy observation.

**PROPOSITION 2.1.** Let  $A$  and  $B$  be abelian groups,  $kA$  and  $kB$  their tensor algebras. Then  $k(A \otimes B) \cong kA \circledast kB$ .

Two other well known phenomena may be elucidated using the tensor and the internal Hom. A one sided analogue of  $\circledast$  allows us to identify the mixed tensor product of a coalgebra  $C$  and an algebra  $A$  [1]. Let  $C \circledast A$  be the free algebra generated by symbols  $c \circledast a$  modulo the usual relations for the tensor product as well as

$$c \circledast aa' = \sum (c_1 \circledast a)(c_2 \circledast a')$$

**PROPOSITION 2.2.**  $\text{Hom}_{\mathcal{A}}(C \circledast A, B) \cong \text{Hom}_{\mathcal{C}}(C, \mathcal{A}(A, B)) \cong \text{Hom}_{\mathcal{A}}(A, C \Rightarrow B)$ .

*Proof.* The second isomorphism may be found in [3] (it is essentially [10, 7.03]), and  $\text{Hom}_{\mathcal{A}}(C \circledast A, B) \cong \text{Hom}_{\mathcal{A}}(A, C \Rightarrow B)$  by computation.  $\diamond$

This mixed tensor completes a hierarchy of tensor-Hom adjointness relations:

$$\begin{aligned} \text{Hom}_{\mathcal{V}}(C \otimes H, A) &\cong \text{Hom}_{\mathcal{C}}(C, \mathcal{V}(H, A)) \\ \text{Hom}_{\mathcal{A}}(C \circledast H, A) &\cong \text{Hom}_{\mathcal{C}}(C, \mathcal{A}(H, A)) \\ \text{Hom}_{\mathcal{A}}(C \circledast H, A) &\cong \text{Hom}_{\mathcal{H}}(C, \mathcal{A}(H, A)) \\ \text{Hom}_{\mathcal{H}}(C \circledast H, A) &\cong \text{Hom}_{\mathcal{H}}(C, \mathcal{H}(H, A)) \end{aligned}$$

Given a commutative algebra  $A$ , its dual coalgebra is  $\mathcal{A}(A, k)$ , denoted  $A^\circ$  in [10]. Of course, the dual of a Hopf algebra  $H$  is again a Hopf algebra, and this may be computed using the internal Hom in  $\mathcal{H}$ . Denote by  $Z^+$  the cofree Hopf algebra over the algebra  $k$ . This may be realized as a sort of constant group scheme:  $Z^+ = k \langle x, {}^1x, {}^2x, \dots \rangle$  where the  ${}^n x$  are orthogonal idempotents and form a sequence of divided powers. Clearly  $G_m^\circ \cong Z^+$  and  $Z^{+\circ} \cong G_m$ .

Then for any Hopf algebra  $H$  we have  $H^\circ \cong \mathcal{A}(H, k) \cong \mathcal{H}(H, Z^+)$  since  $Z^+$  is cofree over  $k$ .

From a categorical point of view our tensor  $\otimes$  is very well behaved. Since  $kZ \cong G_m$ , proposition 2.1 says  $k(-)$  is a monoidal functor from abelian groups to abelian Hopf algebras. Similarly, if  $F : \mathcal{C} \rightarrow \mathcal{H}$  denotes the free Hopf algebra functor [11], then  $kF \cong G_m$  and  $(C \otimes D)F \cong CF \otimes DF$ .

This is important for our understanding of the  $\mathcal{H}$ -enriched cohomology theory introduced in [5]. A triple  $T$  on  $\mathcal{V}$  yields a non-homogeneous complex of coalgebras  $\mathcal{V}(AT^*, B)$  for  $T$ -algebras  $A$  and  $B$ . There is a circle product

$$\circ : \mathcal{V}(AT^m, B) \otimes \mathcal{V}(BT^m, C) \rightarrow \mathcal{V}(AT^{m+n}, C)$$

which obeys the Leibnitz formula [4]. Let  $[AT^*, B] = (\mathcal{V}(AT^*, B))F$ . Since  $\mathcal{H}$  is abelian, the simplicial complex  $[AT^*, B]$  may be collapsed into a chain complex, and we may construct cohomology "groups"  $H[A, B]$ , these being, in fact, abelian Hopf algebras. We see that the circle product lifts to a product

$$\circ : [AT^m, B] \otimes [BT^m, C] \rightarrow [AT^{m+n}, C]$$

which still satisfies the Leibnitz boundary formula, hence may be carried over to homology. Thus  $H^*[A, A]$  becomes a graded ring object in  $\mathcal{C}$ , a "Hopf ring" in the sense of [9], though ours comes not from a Künneth formula but from the algebra structure of  $A$ . The circle product arises naturally in the study of algebraic deformations and their obstructions. As well, it explains the nature of the homology ring of a map [6].

We close with a note on generalization. The tensor  $\otimes$  may equally well be defined for cocommutative bialgebras, and the field  $k$  may be replaced by a commutative ring, the only loss being the abelianness of  $\mathcal{H}$ .

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