

EQUIDECOMPOSABILITY OF SETS, INVARIANT MEASURES, AND PARADOXES (*)

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Contents

1. Introduction
2. Amenable and supramenable groups
3. The Banach-Tarski paradox
4. Decompositions and perfect matchings
5. Local commutativity
6. Paradoxes using contractions and Lipschitz functions
7. The type semigroup
8. A criterion for equidecomposability using translations
9. Two applications: Cavalieri's principle and circle-squaring
10. References

1. Introduction.

The theory of equidecomposable sets was developed by F. Hausdorff, W. Sierpiński, S. Banach, A. Tarski, J. von Neumann and others. After many particular results published mostly in the *Fundamenta Mathematicae* during the years 1914–1924, the first systematic exposition of the theory was given by Banach and Tarski in the paper [3], which also contains the celebrated Banach–Tarski Paradox. Many of the results of this period are of paradoxical nature in that they contradict our geometrical intuition. In fact,

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these results are nothing but spectacular geometric forms of the “paradox of infinity”.

It was observed by Galileo Galilei that an infinite set is equivalent to a proper subset of itself [5]. About 250 years later R. Dedekind realized that this property characterizes the infinite sets and used it as the *definition* of infinity [4].

A simple geometrical interpretation of this phenomenon is provided by the sets \mathbf{N} (non-negative integers) and \mathbf{N}^+ (positive integers); they are not only equivalent, but also congruent, although \mathbf{N}^+ is a proper subset of \mathbf{N} . In the plane we may find bounded sets with this property. Let \mathbf{C} denote the set of complex numbers. Let $c \in \mathbf{C}$ be such that $|c| = 1$ but c is not a root of unity. Then the sets $A = \{c^n : n \in \mathbf{N}\}$ and $B = \{c^n : n \in \mathbf{N}^+\}$ are bounded (subsets of the unit circle), congruent (a rotation maps A onto B), and B is a proper subset of A .

Such an example does not exist in \mathbf{R} . It is easy to see that if A, B are bounded subsets of \mathbf{R} , $A \supset B$ and $A \simeq B$ then $A = B$. ($A \simeq B$ denotes that A and B are congruent.) However, we can find bounded subsets of \mathbf{R} exhibiting the “paradox of infinity” if we replace the notion of congruence by that of equidecomposability. We say that the sets $A, B \subset \mathbf{R}^n$ are *equidecomposable* (or equivalent by finite decomposition) if there are finite partitions $A = \bigcup_{i=1}^k A_i$ and $B = \bigcup_{i=1}^k B_i$ such that $A_i \simeq B_i$ for every $i = 1, \dots, k$. We shall denote this fact by $A \sim B$.

Now let $\alpha \in (0, 1)$ be an irrational number and put $A = \{\{n\alpha\} : n \in \mathbf{N}\}$, $B = \{\{n\alpha\} : n \in \mathbf{N}^+\}$, where $\{x\}$ denotes the fractional part of the number x . Then B is a proper subset of A and yet $A \sim B$. Indeed, denoting $A_1 = A \cap [0, 1 - \alpha)$, $A_2 = A \cap [1 - \alpha, 1)$, $B_1 = B \cap [\alpha, 1)$, $B_2 = B \cap [0, \alpha)$, one can easily check that $B_1 = A_1 + \alpha$ and $B_2 = A_2 + \alpha - 1$. This implies that $[0, 1] \sim (0, 1]$. Indeed, putting $A_3 = B_3 = [0, 1] \setminus A = (0, 1] \setminus B$, we obtain the partitions $[0, 1] = \bigcup_{i=1}^3 A_i$, $(0, 1] = \bigcup_{i=1}^3 B_i$.

A stronger form of the paradox of infinity states that an infinite set can be decomposed into two subsets, each of which is equivalent to the original. In 1914 S. Mazurkiewicz and W. Sierpiński realized that this paradox can also be exhibited geometrically. They constructed a plane set which can be decomposed into two subsets which are congruent to the original. The construction is the following [19]. Let P denote the set of polynomials with non-negative integer coefficients (including the identically 0 polynomial),

and let c be a transcendental complex number with $|c| = 1$. We put $A = \{p(c) : p \in P\}$, $A_1 = A + 1$, $A_2 = cA$. It is easy to prove that $A_1 \cap A_2 = \emptyset$, $A = A_1 \cup A_2$ and $A \simeq A_1 \simeq A_2$.

One can show that a bounded plane set cannot have this property. There are bounded plane sets, however, which have a slightly weaker property. A set $A \subset \mathbf{R}^n$ is called *paradoxical*, if there is a partition $A = A_1 \cup A_2$ such that $A \sim A_1 \sim A_2$. It was proved recently by W. Just [7] that there are bounded plane sets which are paradoxical. (Proof: Let D denote the closed unit disc and let $\varepsilon = (-1 + i\sqrt{3})/2$. Then for every $z \in D$ we can select an $\varepsilon(z) \in \{1, \varepsilon, \varepsilon^2\}$ such that $z + \varepsilon(z) \in D$. We select $\varepsilon(0) = 1$. Let c be a transcendental number with $|c| = 1$. Let B denote the set of numbers $c^k + a_{k-1}c^{k-1} + \dots + a_0$ such that, for every $j = 0, \dots, k-1$ either $a_j = 0$ or $a_j = \varepsilon(c^{k-j} + a_{k-1}c^{k-j-1} + \dots + a_{j+1}c)$. Note that $1 \in B$ (if $k = 0$). We put $A = B \cup \{0\}$; then A is bounded, since $A \subset D$. For $\eta \in \{1, \varepsilon, \varepsilon^2\}$ we put $A_\eta = \{z \in cA : \varepsilon(z) = \eta\}$. One can check, using the fact that c is transcendental, that $A = cA \cup (A_1 + 1) \cup (A_\varepsilon + \varepsilon) \cup (A_{\varepsilon^2} + \varepsilon^2)$ and $cA = A_1 \cup A_\varepsilon \cup A_{\varepsilon^2}$ are partitions, and thus A is paradoxical.)

However, no "reasonable" plane sets can be paradoxical. Banach proved in 1923 [1] that the Lebesgue measure on the plane can be extended to a finitely additive and invariant measure defined on all subsets of \mathbf{R}^2 . That is, there is a map $\mu : P(\mathbf{R}^2) \rightarrow [0, \infty]$ such that $\mu(A) = \lambda_2(A)$ if $A \subset \mathbf{R}^2$ is measurable, and $\mu(A \cup B) = \mu(A) + \mu(B)$ and $\mu(gA) = \mu(A)$ whenever $A, B \subset \mathbf{R}^2$, $A \cap B = \emptyset$ and g is an isometry of \mathbf{R}^2 .

It is easy to see that if $A, B \in \mathbf{R}^2$ and $A \sim B$ then necessarily $\mu(A) = \mu(B)$. Thus, if A is paradoxical, $A = B \cup C$, $B \cap C = \emptyset$, $A \sim B \sim C$, then $\mu(A) = \mu(B) + \mu(C) = 2\mu(A)$ and hence either $\mu(A) = 0$ or $\mu(A) = \infty$. Since μ is an extension of λ_2 , it follows that no measurable plane set with finite, positive measure can be paradoxical.

As John von Neumann realized in [20], the existence of Banach's measure in \mathbf{R}^2 is the consequence of the fact that the group of isometries of \mathbf{R}^2 is solvable. Von Neumann proved that solvable groups are necessarily *amenable*; they support finitely additive, invariant probability measures. The isometry group of \mathbf{R} is also solvable and hence amenable. Sierpiński proved in 1946 that this group has an even stronger property, now called *supramenability*, which implies that in \mathbf{R} there are no paradoxical sets at all.

The situation changes dramatically in \mathbf{R}^3 and in higher dimensions. The isometry groups of these spaces are not solvable, not even amenable; in fact, they contain free subgroups. The consequence of this fact is the celebrated Banach–Tarski paradox: in \mathbf{R}^k ($k \geq 3$) the solid ball is paradox, moreover, any two bounded sets with nonempty interior are equidecomposable.

In the next two sections we shall sketch the proof of these classical results. In Sections 4 and 5 we describe the role of graph theory (perfect matchings of bipartite graphs) and local commutativity in questions of equidecomposability. Section 6 contains paradoxes in \mathbf{R} using contractions (the so-called von Neumann paradox) and, more generally, Lipschitz functions. In Section 7 we discuss measures on semigroups and give a necessary and sufficient condition for the equidecomposability of sets in abstract spaces. Sections 8 and 9 deal with equidecomposability using translations. Here we prove that bounded convex sets of the same positive measure are equidecomposable.

We shall use the following notation. G_n , SG_n , SO_n , T_n will denote the groups of all isometries, orientation-preserving isometries, orthogonal transformations with determinant 1, and translations of \mathbf{R}^n , respectively. Thus SO_2 is the group of rotations of \mathbf{R}^2 about the origin and SO_3 is the group of rotations of \mathbf{R}^3 about axes going through the origin.

The Lebesgue measure in \mathbf{R}^n is denoted by λ_n . The closure, boundary, interior, and cardinality of the set A is denoted by $\text{cl } A$, ∂A , $\text{int } A$ and $|A|$, respectively. The power set of A is denoted by $P(A)$.

We shall say that the group G acts on the set X if G is a group of bijections of X onto itself. The sets $A, B \subset X$ are said to be G -equidecomposable, if there are finite partitions $A = \bigcup_{i=1}^k A_i$ and $B = \bigcup_{i=1}^k B_i$ and transformations $f_i \in G$ ($i = 1, \dots, k$) such that $f_i(A_i) = B_i$ for every $i = 1, \dots, k$. We shall denote this fact by $A \stackrel{G}{\sim} B$. The set $A \subset X$ is G -paradoxical, if there is a partition $A = A_1 \cup A_2$ such that $A \stackrel{G}{\sim} A_1 \stackrel{G}{\sim} A_2$.

2. Amenable and supramenable groups.

Let G be a group. We say that μ is a non-negative finitely additive invariant measure on G if $\mu : P(G) \rightarrow [0, \infty]$, $\mu(A \cup B) = \mu(A) + \mu(B)$

and $\mu(gA) = \mu(A)$ whenever $A, B \subset G$, $A \cap B = \emptyset$ and $g \in G$. G is said to be *amenable*, if there is a non-negative finitely additive invariant measure μ on G such that $\mu(G) = 1$. We say that G is *supramenable*, if for every non-empty $A \subset G$ there is a non-negative finitely additive invariant measure μ on G such that $\mu(A) = 1$.

Our first aim is to show that G_1 is supramenable.

If G is a group and $g_1, \dots, g_r \in G$, then $g_{i_1}g_{i_2} \dots g_{i_n}$ ($i_1, \dots, i_n \in \{1, \dots, r\}$) is called a *word of length n* . The group G is called *exponentially bounded*, if for every $g_1, \dots, g_r \in G$ and $\varepsilon > 0$ there is n_0 such that for every $n > n_0$, the number of different elements of G obtained from g_1, \dots, g_r as a word of length $\leq n$ is at most $(1 + \varepsilon)^n$.

The following two theorems are due to Sierpiński; see [26], pp. 56–58.

THEOREM 2.1. G_1 is exponentially bounded.

Proof. Each isometry of \mathbf{R} is of the form $g(x) = ax + b$ ($x \in \mathbf{R}$), where $a = 1$ or $a = -1$. If $g_1, \dots, g_r \in G_1$, $g_i(x) = a_i x + b_i$ and $g = g_{i_1} \dots g_{i_n}$, then $g(x) = ax + b$, where $a = \pm 1$ and $b = \pm k_1 b_1 \pm \dots \pm k_r b_r$ ($k_i \in \mathbf{N}$, $k_i \leq n$). Thus the number of different b 's is at most $(2n+1)^r$ and the number of different isometries of the form $g_{i_1} \dots g_{i_n}$ is at most $2(2n+1)^r < (1 + \varepsilon)^n$ if n is large enough.

THEOREM 2.2. If an exponentially bounded group G acts on X then X does not contain non-empty G -paradoxical subsets. In particular, \mathbf{R} does not contain non-empty paradoxical subsets.

Proof. Suppose that $\emptyset \neq A \subset X$ is paradoxical, and let $A = B \cup C$, $B \cap C = \emptyset$, $A \stackrel{G}{\sim} B \stackrel{G}{\sim} C$. Then there are partitions $A = \bigcup_{i=1}^r A_i = \bigcup_{j=1}^s A'_j$, $B = \bigcup_{i=1}^r B_i$, $C = \bigcup_{j=1}^s C_j$ and maps $f_i, g_j \in G$ such that $f_i(A_i) = B_i$, $g_j(A'_j) = C_j$ ($i = 1, \dots, r, j = 1, \dots, s$). Let $F_1(x) = f_i(x)$ ($x \in A_i$, $i = 1, \dots, r$), and $F_2(x) = g_j(x)$ ($x \in A'_j$, $j = 1, \dots, s$), then $F_1(A) = B$, $F_2(A) = C$. It is easy to see that the images of a fixed $x \in A$ under the maps

$$F_{i_1} F_{i_2} \dots F_{i_n} \quad (i_1, \dots, i_n = 1, 2)$$

are distinct. Each of these images is of the form $h_1 \dots h_n(x)$, where

$h_1, \dots, h_n \in \{f_1, \dots, f_r, g_1, \dots, g_s\}$. Thus the words of length $\leq n$ with the letters f_i, g_j define at least 2^n different maps of G , which contradicts the fact that G is exponentially bounded.

The following theorem is due to Tarski [27]; we shall prove it in Section 7.

THEOREM 2.3. *Suppose G acts on X and $E \subset X$. Then there is a finitely additive, G -invariant measure $\mu : P(X) \rightarrow [0, \infty]$ with $\mu(E) = 1$ if and only if E is not G -paradoxical.*

THEOREM 2.4. *Consider the following properties of a group G .*

- (i) G is Abelian.
- (ii) G is exponentially bounded.
- (iii) G is supramenable.
- (iv) G is amenable.

Then (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv).

Proof. (i) \Rightarrow (ii) can be proved by the argument of 2.1. Every group may be considered as a group acting on itself, so an application of 2.2 and 2.3 gives (ii) \Rightarrow (iii), while (iii) \Rightarrow (iv) is obvious.

Since there are paradox sets in \mathbf{R}^2 , G_2 cannot be supramenable. Our next aim is to prove that G_2 , or more generally every solvable group, is amenable. To this end we need the notion of integral with respect to finitely additive measures.

Let $\mu : P(X) \rightarrow [0, \infty]$ be finitely additive, let $A \subset X$, $\mu(A) < \infty$, and let $f : A \rightarrow \mathbf{R}$ be bounded. If f has finite range then $f = \sum_{i=1}^k c_i \chi_{A_i}$, where $A = A_1 \cup \dots \cup A_k$ is a partition and then we define $\int_A f d\mu = \sum_{i=1}^k c_i \mu(A_i)$. In general we define $\int_A f = \lim_{n \rightarrow \infty} \int_A f_n d\mu$, where f_n has finite range for every n and $f_n \rightarrow f$ uniformly. It is easy to check that this definition makes sense, and the integral defined in this way has the following properties:

$$\int_A (c_1 f_1 + c_2 f_2) d\mu = c_1 \int_A f_1 d\mu + c_2 \int_A f_2 d\mu, \quad \left| \int_A f d\mu \right| \leq \sup |f| \cdot \mu(A).$$

THEOREM 2.5. *Every solvable group is amenable.*

Proof. It is enough to prove that if G is a normal subgroup of H such that H/G is Abelian and G is amenable, then H is also amenable. By 2.4 every Abelian group is amenable. Therefore, it is enough to show that if G is a normal subgroup of H and if G and H/G are amenable, then so is H .

Let μ and ν be non-negative finitely additive invariant measures on G and H/G , respectively, such that $\mu(G) = \nu(H/G) = 1$. We extend μ by $\mu(hA) = \mu(A)$ ($A \subset G$, $h \in H$). This definition makes sense, since $A_1, A_2 \subset G$, $h_1 A_1 = h_2 A_2$ implies $h_2^{-1} h_1 \in G$ and thus, by the invariance of μ , $\mu(A_1) = \mu(h_2^{-1} h_1 A_1) = \mu(A_2)$. This extension defines μ on the power set of each coset of G .

Let $\phi : H \rightarrow H/G$ be the natural homomorphism, then $\phi^{-1}(y)$ is a coset of G for every $y \in H/G$. Let $A \subset H$ be fixed. Then $g(y) \stackrel{\text{def}}{=} \mu(A \cap \phi^{-1}(y))$ defines a bounded function on H/G . We put $\gamma(A) \stackrel{\text{def}}{=} \int_{H/G} g d\nu$; it is easy to check that γ is an invariant finitely additive measure on H with $\gamma(H) = 1$.

THEOREM 2.6. *Let G be an amenable subgroup of G_n . Then there is a non-negative finitely additive G -invariant extension of λ_n to $P(\mathbb{R}^n)$.*

Proof. One can prove (using, for example, the Hahn–Banach theorem), that λ_n has a non-negative finitely additive extension to $P(\mathbb{R}^n)$. Let ν be such an extension. Let γ be a non-negative finitely additive invariant measure on G with $\gamma(G) = 1$. Then for every $A \subset \mathbb{R}^n$ we define $f_A(g) = \nu(gA)$ ($g \in G$). If f_A is not bounded on G then we define $\mu(A) = \infty$, otherwise we take $\mu(A) \stackrel{\text{def}}{=} \int_G f_A d\gamma$. It is easy to show that μ satisfies the requirements.

COROLLARY 2.7. *There is a non-negative finitely additive G_2 -invariant extension of λ_2 to $P(\mathbb{R}^2)$.*

Proof. G_2 is solvable, because in the sequence of groups {identity}, T_2 , SG_2 , G_2 each group is a normal subgroup of the next, and the factor groups are Abelian.

COROLLARY 2.8. *For every n there is a non-negative finitely additive translation-invariant extension of λ_n to $P(\mathbb{R}^n)$.*

Proof. T_n is Abelian and hence amenable.

3. The Banach–Tarski paradox.

First we prove that SO_3 contains free subgroups. This fact has many proofs (see [30], pp. 15–16). Our proof given below is based on a result on linear fractional transformations which will be used also in one dimensional paradoxes in Section 6.

Let $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the extended complex plane. The linear fractional transformations are the functions $(ax + b)/(cx + d)$ ($a, b, c, d \in \mathbb{C}, ad - bc \neq 0$) mapping $\bar{\mathbb{C}}$ onto itself. The set of linear fractional transformations will be denoted by LFT.

The next theorem is due to John von Neumann [20].

THEOREM 3.1. *Let the complex numbers a_k, b_k, c_k, d_k ($k \in I$) be algebraically independent over the rationals, and let the linear fractional transformations α_k be defined by $\alpha_k(x) = (a_k x + b_k)/(c_k x + d_k)$ ($k \in I$). Then α_k ($k \in I$) generate a free group.*

Proof. We have to prove that if $m_1, \dots, m_r \in \mathbb{Z} \setminus \{0\}$ and $k_i \neq k_{i-1}$ ($i = 2, \dots, r$) then $\alpha = \alpha_{k_1}^{m_1} \dots \alpha_{k_r}^{m_r}$ is not the identity map. It is easy to check that $\alpha(x) = (Ax + B)/(Cx + D)$, where A, B, C, D are polynomials of a_k, b_k, c_k, d_k with integer coefficients, and $AD - BC \neq 0$. If α is the identity map then $B = C = 0$ and $A = D$. Since a_k, b_k, c_k, d_k are algebraically independent, these equations must be identities. This implies that, if n denotes the number of different indices among k_1, \dots, k_r , then for arbitrary $\beta_1, \dots, \beta_n \in \text{LFT}$, $\beta_{k_1}^{m_1} \dots \beta_{k_r}^{m_r}$ is the identity map. Let $\omega(x) = 1/x$ and $\delta(x) = x + 2$. Putting $\beta_k = \delta^k \omega \delta^k$ ($k = 1, \dots, n$), $\beta_{k_1}^{m_1} \dots \beta_{k_r}^{m_r}$ will be of the form $\delta^{n_1} \omega \delta^{n_2} \omega \dots \omega \delta^{n_s}$, where $n_1, \dots, n_s \neq 0$. The value of this map at x is given by the continued fraction

$$2n_1 + \frac{1}{2n_2 + \dots \frac{1}{x + 2n_s}}.$$

If $x \rightarrow \infty$ then this value converges to a finite limit. Therefore this map cannot be the identity.

Let S denote the unit sphere and let $\pi : S \rightarrow \bar{\mathbb{C}}$ denote the stereographic projection. We shall need the following fact of complex function theory: if ρ is a rotation of S then $\pi\rho\pi^{-1} \in LFT$ and is of the form $(ax + b)/(cx + d)$, where $d = \bar{a}$ and $c = -\bar{b}$. Conversely, if $\alpha(x) = (ax + b)/(cx + d)$, where $d = \bar{a}$ and $c = -\bar{b}$, then $\pi^{-1}\alpha\pi$ is a rotation of S . (See [21], p. 55.)

THEOREM 3.2. *Let the real numbers a_k, b_k, c_k, d_k ($k \in I$) be algebraically independent over the rationals, let*

$$A_k = a_k + ib_k, B_k = c_k + id_k, C_k = -c_k + id_k, D_k = a_k - ib_k \quad (k \in I),$$

and let the linear fractional transformations α_k be defined by $\alpha_k(x) = (A_kx + B_k)/(C_kx + D_k)$ ($k \in I$). Then the rotations $\rho_k = \pi^{-1}\alpha_k\pi$ ($k \in I$) generate a free group.

Proof. The numbers A_k, B_k, C_k, D_k ($k \in I$) are also algebraically independent over the rational. Indeed, suppose that there is a finite set $F \subset I$ such that A_k, B_k, C_k, D_k ($k \in F$) are algebraically dependent; then the degree of transcendence of this system (i.e. the cardinality of a maximal algebraically independent subsystem) is less than $4|F|$. Now the numbers a_k, b_k, c_k, d_k ($k \in F$) are algebraically dependent of this system and hence their degree of transcendence would be also less than $4|F|$, which is impossible.

This implies, by 3.1, that the α'_k s are independent. Since $\rho_{k_1}^{m_1} \dots \rho_{k_r}^{m_r} = \pi^{-1}\alpha_{k_1}^{m_1} \dots \alpha_{k_r}^{m_r}\pi$, the ρ'_k s are also independent.

THEOREM 3.3. *There is a subset $E \subset S$ such that (i) S can be covered by four congruent copies of E , and (ii) S contains infinitely many disjoint congruent copies of E .*

Proof. Let a and b be independent rotations of S , and let F denote the group generated by a and b . We define an equivalence relation on S by putting $x \sim y$ if there is an $f \in F$ such that $y = f(x)$. Then each

equivalence class is countable. Let C denote the union of those equivalence classes which contain at least one fixed point of any non-identity element of F . Since F is countable, and each f has exactly two fixed points, C is also countable. Let $H \subset S$ contain exactly one element of each equivalence class. Then every $x \in S \setminus C$ has a unique representation of the form $x = f_x(y)$, where $f_x \in F$ and $y \in H$.

Let U denote the set of those reduced words with letters a and b which begin with a^k ($k \neq 0$). We define

$$E = \{x \in S \setminus C : f_x \in U\}.$$

Then the sets $b^n(E)$ ($n \in \mathbb{N}$) are pairwise disjoint, so that (ii) is satisfied. It is easy to check that $S \setminus C \subset E \cup a(E)$. Since C is countable, there is a rotation c of S such that $C \cap c(C) = \emptyset$. Then $c(C) \subset E \cup a(E)$, $C \subset c^{-1}(E) \cup c^{-1}a(E)$ and hence $S \subset E \cup a(E) \cup c^{-1}(E) \cup c^{-1}a(E)$, which proves (i).

The next result is called the Banach–Schröder–Bernstein theorem (see [2] or [30], p. 25).

THEOREM 3.4. *Let f be an injective map from A into B , and let g be an injective map from B into A . Then there are partitions $A = A_1 \cup A_2$, $B = B_1 \cup B_2$ such that $f(A_1) = B_1$ and $g(B_2) = A_2$.*

Proof. We define a bipartite graph Γ on the pair of sets (A, B) by connecting the points $x \in A$ and $y \in B$ if either $y = f(x)$ or $x = g(y)$. It is easy to check that every connected component of Γ is either a cycle or a path which is infinite in one or both directions. For every connected component we have either $f(C \cap A) = C \cap B$ or $g(C \cap B) = C \cap A$. We define A_1 as the union of $C \cap A$ where C runs through those connected components for which $f(C \cap A) = C \cap B$. We put $A_2 = A \setminus A_1$, $B_1 = f(A_1)$ and $B_2 = g^{-1}(A_2)$.

The following statement is a simple but amusing corollary ([30], pp. 25–26).

COROLLARY 3.5. *Let D be a disc and Q be a square. Then there*

are partitions $D = D_1 \cup D_2$ and $Q = Q_1 \cup Q_2$ such that D_1 is similar to Q_1 and D_2 is similar to Q_2 .

COROLLARY 3.6. *Let G act on X . If $A_1 \subset A \subset X$, $B_1 \subset B \subset X$, $A \stackrel{G}{\sim} B_1$ with n pieces and $A_1 \stackrel{G}{\sim} B$ with k pieces, then $A \stackrel{G}{\sim} B$ with $n + k$ pieces.*

THEOREM 3.7. *The solid ball in \mathbb{R}^3 is paradoxical using 11 pieces.*

Proof. Let E be a set with the properties (i) and (ii) of 3.3, and let $E^* = \cup\{rE : 0 < r \leq 1\}$. Then the unit ball is covered by 5 congruent copies of E^* and also contains 10 disjoint congruent copies of E^* . From this and 3.6 the statement easily follows.

Later we shall see that the number of pieces can be reduced to 5.

THEOREM 3.8. *If A and B are bounded sets in \mathbb{R}^3 with non-empty interior, then $A \sim B$.*

Proof. We may assume that the unit ball is contained in both A and B . Suppose that A and B can be covered by n congruent copies of the unit ball. Since both A and B contain $5n$ disjoint congruent copies of E^* , the statement follows from 3.6.

4. Decompositions and perfect matchings.

We already saw in 3.4 that in the formulation of theorems about mappings the language of graph theory can be useful. Equidecomposability of sets can also be formulated in terms of perfect matchings of bipartite graphs. This formulation was used first by D. König in [9]. Recently W. Just described explicitly this approach in [8]. Although Just only mentions König as his predecessor, Banach and Tarski are also among them (see [28]).

DEFINITION. Let A, B be arbitrary sets. By a bipartite graph on the pair of sets (A, B) we shall mean a multiset of unordered pairs (x, y) such that $x \in A$ and $y \in B$. The pairs (x, y) are called lines connecting x and

y . Note that we allow multiple lines. Also, if $A \cap B \neq \emptyset$ and $x \in A \cap B$, then the loop (x, x) is allowed. We shall use the terms degree, walk, path, cycle and perfect matching in the usual sense (see [18], pp. xxix–xxxii). Thus Γ is a perfect matching, if there is a bijection f of A onto B such that $\Gamma = \{(x, f(x)) : x \in A\}$.

Let G be a family of functions mapping subsets of X into X , and let $A, B \subset X$. We define a bipartite graph on (A, B) by

$$\Gamma_G(A, B) = \{(x, y) : x \in A, y \in B, \exists f \in G, x \in \text{Dom } f, f(x) = y\}.$$

The connection between equidecomposability and perfect matchings is explained by the following theorem. It readily follows from the definitions so we omit the proof.

THEOREM 4.1. *Let G act on X . For every $A, B \in X$ we have $A \stackrel{G}{\sim} B$ if and only if there is a finite subsystem $H \subset G$ such that $\Gamma_H(A, B)$ contains a perfect matching.*

By this theorem, in order to prove the equidecomposability of two sets, we have to find perfect matchings in some bipartite graphs. The most important tool in the search of perfect matchings is the following theorem due to M. Hall [6] and R. Rado [22]. A graph will be called *locally finite*, if the degree of each point is finite. If Y is a subset of the points of a graph Γ then we shall denote by $\Gamma(Y)$ the set of those points of Γ which are connected to at least one point of Y .

THEOREM 4.2. *A locally finite bipartite graph Γ contains a perfect matching if and only if $|\Gamma(Y)| \geq |Y|$ holds for every finite set Y of points of Γ .*

This theorem is an immediate consequence of 3.4 and the following result.

THEOREM 4.3. *Let Γ be a locally finite bipartite graph on the pair of sets (A, B) and suppose that for every finite $Y \subset A$ we have $|\Gamma(Y)| \geq |Y|$. Then there is an injective map $f : A \rightarrow B$ such that $(x, f(x)) \in \Gamma$ for every $x \in A$.*

Proof. If A is finite then this is P. Hall's theorem; a simple proof using induction on $|A|$ can be found in [18], pp. 5–6. Now we turn to the general case. It is enough to define an injective f separately in every connected component of Γ , and thus we may assume that Γ itself is connected. Since Γ is locally finite, this implies that Γ is countable, let $A = \{a_1, a_2, \dots\}$. Let $A_n = \{a_1, \dots, a_n\}$ and let $\Gamma_n = \Gamma \cap (A_n \times B)$. Since Γ_n is finite, it follows from P. Hall's theorem that there is an injective map $f_n : A_n \rightarrow B$ such that $(x, f_n(x)) \in \Gamma$ for every $x \in A_n$. Let V_k denote the set of points connected to a_k . Then V_k is finite for every k , and $f_n(a_k) \in V_k$ for every k and $n \geq k$. This easily implies, by successive selections of infinite subsequences, that there is an injective map $f : A \rightarrow B$ with the required property.

In the sequel we shall give some examples of the applications of Theorem 4.2. The first is a classical result of D. König and S. Valkó [10] and is called “the cancellation law”.

THEOREM 4.4. *Let G act on X . Let $A_1, \dots, A_k, B_1, \dots, B_k \subset X$, and suppose $A_i \cap A_j = B_i \cap B_j = \emptyset$, $A_i \overset{G}{\sim} A_j$, $B_i \overset{G}{\sim} B_j$ ($i \neq j$), and $\bigcup_{i=1}^k A_i \overset{G}{\sim} \bigcup_{i=1}^k B_i$. Then $A_1 \overset{G}{\sim} B_1$.*

Proof. There is a finite subsystem $H \subset G$ such that the graphs $\Gamma_H(A_1, A_i)$, $\Gamma_H(B_1, B_i)$ ($i = 1, \dots, k$) and $\Gamma_H(\bigcup_{i=1}^k A_i, \bigcup_{i=1}^k B_i)$ contain the perfect matchings M_i , N_i ($i = 1, \dots, k$) and P , respectively. We define a graph Γ on (A_1, B_1) as follows: we put (x, y) in Γ if and only if there is an i and there are points $x_i \in A_i$, $y_i \in B_i$ such that $(x, x_i) \in M_i$, $(x_i, y_i) \in P$ and $(y_i, y) \in N_i$. Then $\Gamma \subset \Gamma_{H^3}(A_1, B_1)$ and Γ is regular (the degree of each of its point is k). This easily implies that Γ satisfies the Hall–Rado condition $|\Gamma(Y)| \geq |Y|$ and hence, by 4.2, Γ contains a perfect matching. Obviously, H^3 is a finite subset of G and thus, by 4.1, $A_1 \overset{G}{\sim} B_1$.

We conclude with a simple lemma which will be used in the next section.

LEMMA 4.5. *Let Γ be a connected and locally finite bipartite graph. If the degree of each point of Γ is at least two and if Γ contains at most one cycle, then Γ contains a perfect matching.*

Proof. Let V denote the set of points of Γ . By 4.2, it is enough to show that $|\Gamma(X)| \geq |X|$ holds for every finite $X \subset V$. We shall prove that there is an injective map $f : V \rightarrow V$ such that every point $p \in V$ is adjacent to $f(p)$; this will show that Γ satisfies this condition.

Suppose first that Γ contains no cycle and let a point r be selected. For any $p \in V$, $p \neq r$ there is a unique path $\{r, p_1, p_2, \dots, p_n = p\}$ connecting r and p . Since the degree of p is at least two, we can select a point $f(p)$ such that $f(p)$ is adjacent to p and $f(p) \neq p_{n-1}$. We take any point q adjacent to r and put $f(r) = q$. It is easy to check that f is injective.

Now suppose that Γ contains a cycle $C = \{q_0, q_1, \dots, q_{k-1}, q_k = q_0\}$. We define $f(q_i) = q_{i+1}$ ($i = 0, 1, \dots, k-1$). If $p \notin C$ then there is a unique path $\{p_0, p_1, \dots, p_n = p\}$ such that $p_0 \in C$ and $p_i \notin C$ ($i = 1, \dots, n$). Since the degree of p is at least two, there is a point $f(p)$ which is adjacent to p and is different from p_{n-1} . In this way we have defined an injective map, and this completes the proof.

5. Local commutativity.

Let G act on X . We say that G is *locally commutative* provided that whenever two elements of G have a common fixed point then they commute. The role of local commutativity in the theory of equidecomposability was discovered by R.M. Robinson in [23]. In this paper he finds the minimal number of pieces which are needed to duplicate a ball. Banach and Tarski in their paper [3] did not specify the number of pieces to obtain a paradoxical decomposition of the ball. In 1929 von Neumann remarked that 9 pieces suffice. Sierpiński used 8 pieces in [25]. Finally, Robinson showed in [23] that the minimal number is 5. He proved that S is paradoxical using four pieces; his proof is based on the fact that the group of the rotations of a sphere is locally commutative: if two rotations have a common fixed point then they have the same axis and hence they commute. As for the applications of local commutativity in questions of equidecom-

posability, see [30], Chapter 4. Here we shall prove Robinson's theorem through the following result taken from [17].

THEOREM 5.1. *Let G be a locally commutative group acting on a set X and suppose that G is freely generated by the transformations f_1, \dots, f_n . Let A, B, H_1, \dots, H_n be subsets of X such that*

- (i) *for every $x \in A$ there are indices $1 \leq i, j \leq n, i \neq j$ such that $x \in H_i \cap H_j$ and $f_i(x) \in B, f_j(x) \in B$; and*
- (ii) *for every $y \in B$ there are indices $1 \leq i, j \leq n, i \neq j$ and points $x_i \in H_i \cap A, x_j \in H_j \cap A$ such that $f_i(x_i) = f_j(x_j) = y$.*

Then there are partitions $A = \bigcup_{i=1}^n A_i$ and $B = \bigcup_{i=1}^n B_i$ such that $A_i \subset H_i$ and $f_i(A_i) = B_i$ for every $i = 1, \dots, n$.

Sketch of proof. Let $F = \{f_i|H_i : i = 1, \dots, n\}$ and $\Gamma = \Gamma_F(A, B)$. We have to prove that Γ contains a perfect matching. In order to show this, it is enough to prove that there is one in any connected component Γ_1 of Γ or, that Γ_1 satisfies the conditions of 4.5. Since the degree of each point of Γ is at most n , Γ_1 is locally finite. The conditions (i) and (ii) of Theorem 5.1 imply that the degree of each point of Γ_1 is at least two. Finally, the fact that Γ_1 contains at most one cycle, can be deduced from the local commutativity of G . For the details we refer to [17].

THEOREM 5.2. *S is paradoxical using 4 pieces.*

Proof. Let $S_1 = S \times \{1\}$ and $S_2 = S \times \{2\}$. Let g_i ($i = 1, \dots, 4$) be independent rotations of S , and define f_i ($i = 1, \dots, 4$) on $X = S_1 \cup S_2$ by

$$f_i(x, j) = (g_i(x), j) \quad (x \in S, i = 1, 2, j = 1, 2), \text{ and}$$

$$f_i(x, j) = (g_i(x), 3 - j) \quad (x \in S, i = 3, 4, j = 1, 2).$$

It is easy to check that f_i ($i = 1, \dots, 4$) generate a free and locally commutative group on X . Now we apply 5.1 with $A = X, B = S_1, H_1 = H_2 = S_1, H_3 = H_4 = S_2$. Thus we obtain partitions $S_1 = A_1 \cup A_2$ and $S_2 = A_3 \cup A_4$ such that $f_1(A_1) \cup f_2(A_2) \cup f_3(A_3) \cup f_4(A_4)$ is a partition of S_1 . Taking the projections to S we obtain partitions $S = B_1 \cup B_2$ and

$S = B_3 \cup B_4$ such that $g_1(B_1) \cup g_2(B_2) \cup g_3(B_3) \cup g_4(B_4)$ is a partition of S .

Denoting $H^* = \cup\{rH : 0 < r \leq 1\}$ for every $H \subset S$, we have $S^* = B \setminus \{0\}$, where B is the unit ball. Thus $S^* = B_1^* \cup B_2^* = B_3^* \cup B_4^* = g_1(B_1^*) \cup g_2(B_2^*) \cup g_3(B_3^*) \cup g_4(B_4^*)$ gives a duplication of S^* using four pieces. This easily implies that $B \cup B'$, the union of two disjoint copies of B , is equidecomposable to the union of B and a singleton using five pieces. With some more work one can show that $B \cup B' \sim B$ using five pieces (see [30], Theorem 4.7, p. 40).

6. Paradoxes using contractions and Lipschitz functions.

A function f defined on $A \subset \mathbb{R}$ is a *contraction*, if there is a $q < 1$ such that $|f(x) - f(y)| \leq q|x - y|$ holds for every $x, y \in A$. A map $f : A \rightarrow \mathbb{R}$ is called *piecewise contractive* if there is a finite partition $A = A_1 \cup \dots \cup A_n$ such that the restriction $f|_{A_i}$ is a contraction for every $i = 1, \dots, n$. The following theorem was proved by von Neumann in [20].

THEOREM 6.1. *For arbitrary intervals I and J there is a piecewise contractive bijection from I onto J .*

Proof. Let I_1 and I_2 be disjoint subintervals of I . Let c_1, \dots, c_n be real numbers such that, for every $x \in I_1$, $\frac{1}{2}x + c_i \in \text{int } J$ holds for at least two distinct i 's, and, for every $y \in J$, $2(y - c_i) \in \text{int } I_1$ holds for at least two distinct i 's. Let $f_1, \dots, f_n \in LFT$ be chosen such that they have algebraically independent coefficients, and, for each i , f_i approximates the function $\frac{1}{2}x + c_i$ on I_1 so well that (i) $f_i|_{I_1}$ is a contraction, (ii) for every $x \in I_1$ $f_i(x) \in J$ for at least two distinct i 's and (iii) for every $y \in J$ $f_i^{-1}(y) \in I_1$ for at least two distinct i 's.

Let $G = \{f_1, \dots, f_n\}$ and $\Gamma = \Gamma_G(I_1, J)$, then the degree of each point of Γ is at least two. Let F be the group generated by f_1, \dots, f_n ; then, by 3.1, F is free. Let C be the union of those connected components of Γ which contain at least one fixed point of a non-identity element of F ; then C is countable. If $y \in J \setminus C$ then the connected component of Γ containing y does not contain a cycle (because a cycle would give a non-empty word

with a fixed point in the component). Thus by 4.5, this component contains a perfect matching. This is true for every component disjoint from C and hence $\Gamma_G(I_1 \setminus C, J \setminus C)$ contains a matching. This implies that there is a bijection f from $I_1 \setminus C$ onto $J \setminus C$ which is piecewise contractive (since it consists of restrictions of f_1, \dots, f_n). Since $J \cap C$ is countable, it is easy to prove that there is an injective map $g : J \cap C \rightarrow I_2$ such that g^{-1} is piecewise contractive. Then the map g_1 defined by $g_1(x) = f^{-1}(x)$ ($x \in J \setminus C$), $g_1(x) = g(x)$ ($x \in J \cap C$) is an injective map from J into I . Let h be a contraction from I into J . By 3.4 there are partitions $I = A_1 \cup A_2$, $J = B_1 \cup B_2$ such that $h(A_1) = B_1$ and $g_1(B_2) = A_2$. Then the function h_1 defined by $h_1(x) = h(x)$ ($x \in A_1$), $h_1(x) = g_1^{-1}(x)$ ($x \in A_2$) is a piecewise contractive bijection from I onto J .

COROLLARY 6.2. *If A, B are bounded subsets of \mathbf{R} with nonempty interior, then A can be mapped, using a piecewise contractive map, onto B .*

Proof. Let I and J be intervals such that $I \subset A$ and $B \subset J$. Let ϕ be an injective contraction mapping A into $\text{int } B$. By 6.1, there is a piecewise contractive bijection ψ from I onto J . Let ψ_0 denote the restriction of ψ to $\psi^{-1}(B)$, then ψ_0^{-1} is an injective map from B into A . By 3.4, there are partitions $A = A_1 \cup A_2$, $B = B_1 \cup B_2$ such that $\phi(A_1) = B_1$ and $\psi_0(A_2) = B_2$. Thus the map f defined by $f(x) = \phi(x)$ ($x \in A_1$), $f(x) = \psi_0(x)$ ($x \in A_2$) is a piecewise contractive bijection from A onto B .

Suppose that A can be mapped, using a piecewise contractive map, onto B . If $\lambda(A)$ and the number of pieces, n , are given then $\lambda(B)$ cannot exceed $n\lambda(A)$. In the next theorem [17] we give the sharper estimate $\lambda(B) \leq n\lambda(A)/2$, and also show that this bound is the best possible.

THEOREM 6.3.

- (i) *Let $A, B \subset \mathbf{R}$ be measurable and suppose that there is a map $f : A \rightarrow \mathbf{R}$ and a partition $A = A_1 \cup \dots \cup A_n$ such that $B = f(A)$ and $f|_{A_i}$ is a contraction for every $i = 1, \dots, n$. Then $\lambda(B) < n \cdot \lambda(A)/2$.*
- (ii) *Let $A \subset \mathbf{R}$ be measurable and let J be an interval with $|J| < n \cdot \lambda(A)/2$, where n is a positive integer. Then there is a map $f : A \rightarrow$*

\mathbf{R} and a partition $A = A_1 \cup \dots \cup A_n$ such that $f(A) = J$ and $f|_{A_i}$ is a contraction for every $i = 1, \dots, n$. If A is an interval, then f can be chosen to be a bijection between A and J .

Let J be an interval with $1 < |J| < 3/2$. Then (ii) of Theorem 6.3 implies that there is a bijection from $[0,1]$ onto J which consists of three contractions; that is, von Neumann's paradox can be realized using three pieces. On the other hand, such a paradoxical decomposition does not exist if only two pieces can be used, as (i) of 6.3 shows.

We shall denote the Lipschitz constant of a function $f : A \rightarrow \mathbf{R}$ by

$$\text{Lip } f = \sup \{ |(f(x) - f(y)) / (x - y)| : x, y \in A, x \neq y \}.$$

In [12], Theorem 4 we proved the following.

Let $A = A_1 \cup \dots \cup A_n$ be a partition of the set $A \subset \mathbf{R}$ and let $f : A \rightarrow \mathbf{R}$ be a map such that $f|_{A_i}$ is a Lipschitz function with $\text{Lip}(f|_{A_i}) \leq M_i$ for every $i = 1, \dots, n$. Then the inner Lebesgue measure of $f(A)$ is at most $M \cdot \lambda(A)$, where

$$(1) \quad M = \max \left(M_1, \dots, M_n, \frac{1}{2} \sum_{i=1}^n M_i \right).$$

If the restrictions $f|_{A_i}$ are contractions, then we may apply this estimate with $M_1 = \dots = M_n = 1 - \varepsilon$, and obtain (i) of Theorem 6.3. The next theorem shows that the estimate given above is sharp. It also implies (ii) of Theorem 6.3 by putting $M_1 = \dots = M_n = 1 - \varepsilon$.

THEOREM 6.4. *Let M_1, \dots, M_n be positive numbers and let M be defined by (1). Then for every measurable set $A \subset \mathbf{R}$ and for every $0 < d < M \cdot \lambda(A)$ there is a partition $A = A_1 \cup \dots \cup A_n$ and there is a function $f : A \rightarrow \mathbf{R}$ such that $f|_{A_i}$ is a Lipschitz function with $\text{Lip}(f|_{A_i}) \leq M_i$ for every $i = 1, \dots, n$ and $f(A)$ is an interval of length d .*

If A is an interval, then f can be chosen to be a bijection.

Sketch of the proof. Suppose first that A is an interval, say $A = [0, 1]$. First we need the local commutativity of some groups of linear fractional transformations.

LEMMA 6.5. *Let the real numbers a_k, b_k, c_k, d_k ($k = 1, \dots, n$) be algebraically independent over the rationals, let the linear fractional transformations α_k be defined by $\alpha_k(x) = (a_k x + b_k)/(c_k x + d_k)$ ($k = 1, \dots, n$), and let G be the group generated by $\alpha_1, \dots, \alpha_n$. Then G is locally commutative.*

We apply a modified version of the proof of 6.1, where we replace the linear functions $(x/2) + c_i$ by the functions $l_i(x) = (M_i - \varepsilon)x + c_i$ ($x \in [0, 1/2]$) and $r_i(x) = (M_i - \varepsilon)(1 - x) + c_i$ ($x \in [1/2, 1]$). As in the proof of 6.1, we approximate these functions by linear fractional transformation with algebraically independent coefficients, and take the generated group G . This group is free and locally commutative and hence an application of 5.1 yields the statement. For the details we refer to [17].

Next suppose that $A \subset \mathbf{R}$ is measurable and let J be an interval of length $d < M \cdot \lambda(A)$. Let K be a compact subset of A such that $d < M \cdot \lambda(K)$, and let $I = [0, \lambda(K)]$. Since I is an interval, there is a partition $I = C_1 \cup \dots \cup C_n$ and there is a function $g : I \rightarrow \mathbf{R}$ such that $g(I) = J$ and $\text{Lip}(g|_{C_i}) \leq M_i$ for every $i = 1, \dots, n$.

Let $h(x) = \lambda(K \cap (-\infty, x])$ ($x \in A$), then $\text{Lip } h \leq 1$. Since K is compact, it is easy to see that $h(K) = I$. Thus $h(K) \subset h(A) \subset I$ implies that h maps A onto I . Therefore the sets $A_i = h^{-1}(C_i)$ ($i = 1, \dots, n$) and the function $f = g \circ h$ satisfy the requirements of Theorem 6.4.

We mention some problems concerning the higher dimensional analogues of the previous results. Let $A \subset \mathbf{R}^k$ be measurable, and let $f : A \rightarrow \mathbf{R}^k$ be a map such that $\text{Lip}(f|_{A_i}) \leq M_i$ for every $i = 1, \dots, n$. Then the inner Lebesgue measure of $f(A)$ is at most $M \cdot \lambda_k(A)$, where

$$(2) \quad M = \max \left(M_1^k, \dots, M_n^k, \frac{1}{2} \sum_{i=1}^n M_i^k \right)$$

(see [12], Theorem 4). This implies that (i) of Theorem 6.3 remains valid in every dimension. We do not know, however, whether or not (ii) of Theorem 6.3 remains valid in \mathbf{R}^k . It is easy to see that the value of M given by (2) is sharp; for every measurable $A \subset \mathbf{R}^k$ and $d < M \cdot \lambda_k(A)$ there is a function $f : A \rightarrow \mathbf{R}^k$ and a partition $A = A_1 \cup \dots \cup A_n$ such that Lip

$f|A_i \leq M_i$, for every $i = 1, \dots, n$ and $f(A)$ is a measurable set of measure d . The problem is that we cannot ensure that $f(A)$ is an interval. (Even if A is an interval, the proof only gives a set $f(A)$ which is a finite union of intervals.) Therefore the following problem remains open.

PROBLEM 1. Let $I \subset \mathbb{R}^k$ be an interval and let M_1, \dots, M_n be given positive numbers. What is the supremum of the measures of those intervals J for which there is a function $f : I \rightarrow \mathbb{R}^k$ and a partition $I = A_1 \cup \dots \cup A_n$ such that $\text{Lip } f|A_i \leq M_i$ for every $i = 1, \dots, n$ and $f(I) = J$?

Of course, we may ask the same question for every measurable $A \subset \mathbb{R}^k$ instead of an interval I . In this case, however, we do not know even the existence of a piecewise Lipschitz map of the set A onto an interval. (As we saw above, in \mathbb{R} every measurable set of positive measure can be mapped, using a Lipschitz function, onto an interval.) Therefore we face the following question.

PROBLEM 2 * . Let $A \subset \mathbb{R}^k$ be a measurable set of positive measure. Does there exist a Lipschitz map $f : A \rightarrow \mathbb{R}^k$ such that $f(A)$ is an interval?

We mention another result in connection with von Neumann's paradox [12].

THEOREM 6.6. *Let I, J be intervals with $|J| < 2|I|$. Then there are partitions $I = \cup_{i=1}^4 A_i$ and $J = \cup_{i=1}^4 B_i$, and there is a contraction f such that $f(A_1) = B_1$ and B_i is a translated copy of A_i for $i = 2, 3, 4$.*

One can also show that for $|J| \geq 2|I|$ such a paradoxical decomposition does not exist, even if we use more parts to be translated (see [12]).

7. The type semigroup.

The type semigroup was invented by Tarski in [27]; it is formed by the same kind of abstraction as the notion of cardinal and ordinal numbers.

(*) Added in proof: Recently David Preiss proved that for $k = 2$ the answer is affirmative. For $k \geq 3$ the problem is still open.

Let G act on X . Roughly speaking, the elements of the type semigroup are the equivalence classes under the equivalence relation \sim^G and if a, b are the classes containing the sets A and B , respectively, then $a + b$ is defined as the class containing the set $A' \cup B'$, where A', B' are "disjoint copies" of A and B . However, as disjoint copies of A and B not necessarily exist in X , we have to enlarge X and the action G as follows. Let $X^* = X \times \mathbb{N}$ and let R be the ring of sets $\bigcup_{i=0}^n (A_i \times \{i\})$ ($n \in \mathbb{N}$, $A_i \subset X$, $i = 0, \dots, n$). If $g \in G$ and π is a permutation of \mathbb{N} then we define the map (g, π) by

$$(g, \pi)(x, n) = (g(x), \pi(n)) \quad (x \in X, n \in \mathbb{N}).$$

Obviously, the set G^* of all these maps (g, π) forms a group of bijections of X^* onto itself, and R is a G^* -invariant ring. The semigroup types are defined as the equivalence classes with respect to the equivalence relation \sim^* in R . If $[A]$ and $[B]$ denote the classes containing the sets $A, B \in R$, respectively, then $[A] + [B]$ is defined as the class containing the set $g_1(A) \cup g_2(B)$, where $g_1, g_2 \in G^*$ are such that $g_1(A) \cap g_2(B) = \emptyset$. It is easy to check that this operation is well-defined and makes the set of types a commutative semigroup denoted by S . If we identify X with $X \times \{0\}$ then for every $A, B \subset X$ we have $A \sim^G B$ if and only if $A \sim^* B$. That is, $[A] = [B]$ if and only if $A \sim^G B$.

In the language of the type semigroup the cancellation law 4.4 simply becomes the statement $na = nb \Rightarrow a = b$ ($a, b \in S$). Also, Tarski's theorem 2.3 is equivalent to the following statement.

THEOREM 7.1. *If $a \in S$ and $a \neq 2a$, then there is a homomorphism ϕ of S into the additive semigroup $[0, \infty]$ such that $\phi(a) = 1$.*

Sketch of the proof. Using the cancellation law and $a \neq 2a$ one can show that the elements na ($n \in \mathbb{N}$) are distinct. Let $S_a = \{na : n \in \mathbb{N}\}$ and $F = \{x \in S : x + y = na \text{ for some } y \in S, n \in \mathbb{N}\}$. Then S_a, F are subsemigroups of S . Let $\phi(na) = n$ ($n \in \mathbb{N}$), then ϕ has the property that whenever $x_1, \dots, x_n, y_1, \dots, y_k \in S_a$ and $x_1 + \dots + x_n + z = y_1 + \dots + y_k$ for some $z \in F$ then $\phi(x_1) + \dots + \phi(x_n) \leq \phi(y_1) + \dots + \phi(y_k)$. Then one proves, using transfinite induction, that ϕ can be extended to F preserving this property. Finally, we define $\phi(x) = \infty$ for every $x \in S \setminus F$.

Our next aim is to give a necessary and sufficient condition for the equidecomposability of sets [14]. Obviously, if $A \stackrel{G}{\sim} B$ then $\mu(A) = \mu(B)$ holds for every G -invariant finitely additive measure μ . However, this condition is not sufficient for $A \stackrel{G}{\sim} B$, as the following simple example shows.

Let $X = \mathbf{Q}$ be the set of rationals, and let G denote the group of all translations by rational numbers. Let μ be any G -invariant finitely additive measure on \mathbf{Q} . If $\mu([0, 1) \cap \mathbf{Q}) = \infty$ then obviously $\mu([0, 1] \cap \mathbf{Q}) = \infty$. If $\mu([0, 1) \cap \mathbf{Q}) < \infty$, then $\mu(\{x\}) = 0$ for every $x \in \mathbf{Q}$ and hence $\mu([0, 1] \cap \mathbf{Q}) = \mu([0, 1) \cap \mathbf{Q})$. That is $\mu([0, 1] \cap \mathbf{Q}) = \mu([0, 1) \cap \mathbf{Q})$ holds for every G -invariant finitely additive measure μ . On the other hand, it is easy to see that $[0, 1] \cap \mathbf{Q}$ and $[0, 1) \cap \mathbf{Q}$ are not equidecomposable (see [26], Theorem 17, p. 48).

We shall prove that if $\eta(A) = \eta(B)$ holds for every G -invariant finitely additive *signed* measure, then necessarily $A \stackrel{G}{\sim} B$. However, in this criterion we cannot restrict η to finite valued signed measures. Indeed, let $X = \mathbf{Z}$ and let G be the group of translations of \mathbf{Z} by integers. If η is a finite valued G -invariant signed measure on \mathbf{Z} then $\eta(\mathbf{N}) = \eta(\mathbf{N}^+)$ and hence $\eta(\{0\}) = 0 = \eta(\emptyset)$. On the other hand, $\{0\}$ and \emptyset are not equidecomposable.

Therefore, by a G -invariant finitely additive signed measure we shall mean a map η from the subsets of X into $\mathbf{R} \cup \{\infty\}$ such that

$$(1) \quad \eta(g(A)) = \eta(A) \quad (A \subset X, g \in G),$$

and

$$(2) \quad \eta(A \cup B) = \eta(A) + \eta(B) \quad (A, B \subset X, A \cap B = \emptyset),$$

where we adopt the convention

$$\infty + \infty = \infty + a = \infty \quad (a \in \mathbf{R}).$$

THEOREM 7.2. *For every $A, B \subset X$ we have $A \stackrel{G}{\sim} B$ if and only if $\eta(A) = \eta(B)$ holds for every G -invariant finitely additive signed measure η .*

We shall also consider the more general situation where the sets A, B and the pieces used in the decompositions are restricted to be in a prescribed ring of subsets of X .

By a *space* we shall mean a triple (X, G, \mathcal{A}) , where A is a non-empty set, G is a group of bijections of X onto itself, and \mathcal{A} is a G -invariant ring of subsets of X . We say that the sets $A, B \in \mathcal{A}$ are G -equidecomposable in \mathcal{A} , if they are G -equidecomposable in such a way that the pieces used in the decompositions belong to \mathcal{A} . A map $\eta : \mathcal{A} \rightarrow \mathbf{R} \cup \{\infty\}$ is said to be a G -invariant finitely additive signed measure on \mathcal{A} , if (1) and (2) hold with A, B restricted to be elements of \mathcal{A} .

Our aim is to characterize those spaces in which the conditions (i) $A \stackrel{G}{\sim} B$ in \mathcal{A} ; and (ii) $\eta(A) = \eta(B)$ whenever η is a G -invariant finitely additive signed measure on \mathcal{A} are equivalent. Obviously (i) \Rightarrow (ii) in every space. We shall prove that (ii) \Rightarrow (i) if and only if the cancellation law holds in (X, G, \mathcal{A}) .

We shall need the type semigroup of the space (X, G, \mathcal{A}) ; its definition can be obtained from the definition given in the beginning of this section, with the obvious modifications. Let $[A]$ denote the type of $A \in \mathcal{A}$.

THEOREM 7.3. *Let (X, G, \mathcal{A}) be a space and let $A, B \in \mathcal{A}$ be arbitrary. Then $\eta(A) = \eta(B)$ holds for every G -invariant finitely additive signed measure on \mathcal{A} if and only if there is a positive integer n such that $n[A] = n[B]$.*

We say that the cancellation law holds in the space (X, G, \mathcal{A}) if for every $A, B \in \mathcal{A}$ and $n \in \mathbf{N}^+$, $n[A] = n[B]$ implies $[A] = [B]$. By Theorem 7.3, if the cancellation law holds in (X, G, \mathcal{A}) , then conditions (i) and (ii) above are equivalent.

On the other hand, if the cancellation law does not hold, then (ii) does not imply (i). Indeed, let $A, B \in \mathcal{A}$ and $n \in \mathbf{N}^+$ be such that $n[A] = n[B]$ but $[A] \neq [B]$. It is easy to see that $n[A] = n[B]$ implies (ii) and hence (ii) \Rightarrow (i) does not hold. Thus we obtain the following result.

COROLLARY 7.4. *In every space, conditions (i) and (ii) are equivalent if and only if the cancellation law holds.*

Since the cancellation law holds if $\mathcal{A} = P(X)$, this implies Theorem 7.2. If ϕ is a homomorphism from the type semigroup into the additive semigroup $\mathbf{R} \cup \{\infty\}$ then

$$\eta(A) \stackrel{\text{def}}{=} \phi([A]) \quad (A \in \mathcal{A})$$

defines a G -invariant finitely additive signed measure on \mathcal{A} . Therefore the statement of Theorem 7.3 is an immediate consequence of the following result.

THEOREM 7.5. *Let $(S, +)$ be a commutative semigroup, and let $a, b \in S$ be such that $\phi(a) = \phi(b)$ for every homomorphism $\phi : S \rightarrow \mathbf{R} \cup \{\infty\}$. Then there is a positive integer n such that $na = nb$.*

Sketch of the proof. Let the relation Θ be defined by $x\Theta y$ if $nx = ny$ for some $n \in \mathbf{N}^+$. It is easy to see that Θ is a congruence; that is, Θ is an equivalence relation on S such that $x\Theta y$ implies $(x + z)\Theta(y + z)$ for every $z \in S$. Let $S_1 = S/\Theta$ be the factor semigroup and let ψ be the natural homomorphism from S into S_1 . Obviously, the cancellation law holds in S_1 ; i.e. if $x, y \in S_1$ and $nx = ny$ for some $n \in \mathbf{N}^+$ then $x = y$. If ϕ is any homomorphism from S_1 into $\mathbf{R} \cup \{\infty\}$ then the composition of ψ and ϕ will be a homomorphism from S into $\mathbf{R} \cup \{\infty\}$.

Therefore, replacing S by S_1 , if necessary, we may assume that the cancellation law holds in S . We have to prove that, under this condition, distinct elements of S can be separated by homomorphisms mapping into $\mathbf{R} \cup \{\infty\}$.

We define the relation \leq on S by putting $x \leq y$ if there is a $z \in S$ and $n \in \mathbf{N}^+$ such that $x + z = ny$. (This relation is transitive and reflexive but, in general, is not antisymmetric.) The proof of 7.5 is based on the following lemmas (for the proofs, see [14]).

LEMMA 7.6. (i) *If $x, y, z \in S$, $n \in \mathbf{N}^+$ and $x + nz = y + nz$, then $x + z = y + z$.* (ii) *If $x, y, z \in S$, $z \leq x$, $z \leq y$ and $x + z = y + z$, then $x = y$.*

We shall say that a subsemigroup $G \subset S$ is *dense* in S if for every $x \in S$ there is a $y \in G$ such that $x + y \in G$.

LEMMA 7.7. *If G is a dense subsemigroup of S and ϕ is a homomorphism from G into the additive semigroup of the reals then ϕ can be extended to S as a homomorphism.*

LEMMA 7.8. *Let C denote the semigroup generated by the distinct elements $a, b \in S$. If $a \leq b$ and $b \leq a$ then there is a homomorphism $\phi : C \rightarrow (\mathbf{R}, +)$ such that $\phi(a) \neq \phi(b)$.*

Then 7.5 can be proved as follows. We have to show that if $a \neq b$ then there is a homomorphism ϕ of S into $\mathbf{R} \cup \{\infty\}$ which separates a and b (recall that the cancellation law holds in S).

Suppose first that $a \leq b$ does not hold. Then

$$\phi(x) = \begin{cases} 0, & \text{if } x \leq b, \\ \infty, & \text{otherwise} \end{cases}$$

defines a homomorphism such that $\phi(b) = 0$ and $\phi(a) = \infty$. If $b \leq a$ does not hold then we can find a separating homomorphism in the same way. Therefore we may assume that $a \neq b$, $a \leq b$ and $b \leq a$. Then, by Lemma 7.8, there is a homomorphism ϕ of C into \mathbf{R} which separates a and b . Let $S' = \{x \in S : x \leq a\}$. Then S' is a subsemigroup of S in which C is dense. By Lemma 7.7, we can extend ϕ to S' . Finally we extend ϕ from S' to S by putting $\phi(x) = \infty$ for every $x \in S \setminus S'$. It is easy to check that ϕ is a homomorphism into $\mathbf{R} \cup \{\infty\}$, and this completes the proof.

We remark that the statements of Lemma 7.6 together with the implication

$$(3) \quad nx + kz = ny + kz \ (n, k \in \mathbf{N}^+) \Rightarrow x + z = y + z$$

were proved by Tarski in the special case when S is the semigroup of equidecomposability types with unrestricted pieces (see [28], pp. 221–222 and [29], Theorem 16.9, p. 223). Tarski proved these statements by generalizing König's proof of the cancellation law (see note 9 at the end of [28]). As Lemma 7.6 shows, these assertions are direct algebraic consequences of the cancellation law. As for (3), we can argue as follows. By (i) of Lemma 7.6, the condition of (3) implies $nx + z = ny + z$. This gives $n(x + z) = (nx + z) + (n - 1)z = (ny + z) + (n - 1)z = n(y + z)$ and hence we have $x + z = y + z$ by the cancellation law.

8. A criterion for equidecomposability using translations.

Our aim is to give a sufficient condition for $A \overset{T_k}{\sim} B$ ($A, B \subset \mathbb{R}^k$). We shall give this condition in terms of the discrepancy of some special sequences [16].

Let I^k denote the unit cube $\{(x_1, \dots, x_k) : 0 \leq x_i < 1 \ (i = 1, \dots, k)\}$. If $F \subset I^k$ is finite, $|F| = N$, and $H \subset I^k$ is measurable, then the discrepancy of F with respect to H is defined as

$$D(F; H) = \left| \frac{1}{N} |F \cap H| - \lambda_k(H) \right|.$$

The (absolute) discrepancy of the finite set $F \subset I^k$ is defined as

$$D(F) = \sup_J D(F; J),$$

where the sup is taken over all subintervals $J \subset I^k$.

If $a \in \mathbb{R}$ then $\{a\}$ denotes the fractional part of a , that is, $\{a\} = a - [a]$. For every $z = (z_1, \dots, z_k) \in \mathbb{R}^k$ we denote $(z) = (\{z_1\}, \dots, \{z_k\})$ (i.e. $(z) \in I^k$ and $z - (z) \in \mathbb{Z}^k$). If $u, x_1, \dots, x_d \in \mathbb{R}^k$ and N is a positive integer, then we put

$$F_N(u; x_1, \dots, x_d) = \{(u + n_1 x_1 + \dots + n_d x_d) : n_i = 0, \dots, N-1 \ (i = 1, \dots, d)\}.$$

THEOREM 8.1. *Let H_1, H_2 be measurable subsets of I^k with $\lambda_k(H_1) = \lambda_k(H_2) > 0$ and suppose that there are vectors $x_1, \dots, x_d \in \mathbb{R}^k$ such that*

- (i) *the unit vectors $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ ($i = 1, \dots, k$) and x_1, \dots, x_d are linearly independent over the rational numbers, and*
- (ii) *there are positive constants C, ε such that*

$$D(F_N(u; x_1, \dots, x_d); H_r) \leq C \cdot N^{-1-\varepsilon}$$

for every $u \in \mathbb{R}^k$, $N = 1, 2, \dots$, and $r = 1, 2$. Then $H_1 \overset{T_k}{\sim} H_2$.

We shall need a combinatorial result ([15], Remark 3.3), which we present here without proof.

A *lattice cube* is a set of the form $Q = \prod_{i=1}^d [a_i, a_i + n)$ where $a_i \in \mathbf{Z}$ ($i = 1, \dots, d$) and n is a positive integer. The length of the side of a cube Q is denoted by $s(Q)$. The family of all sets which are finite unions of unit cubes will be denoted by \mathcal{H}^d . Obviously, every lattice cube belongs to $\mathcal{H}^{(d)}$. If $H \in \mathcal{H}^{(d)}$, then $p(H)$ will denote the surface area of H (i.e. the $d - 1$ dimensional Hausdorff measure of ∂H). For every $k \in \mathbf{N}$ we put

$$D_k^{(d)} = \left\{ \prod_{i=1}^d [a_i \cdot 2^k, (a_i + 1)2^k) : a_i \in \mathbf{Z}, i = 1, \dots, d \right\}.$$

Thus $D_0^{(d)}$ is the set of unit cubes. The system of *dyadic cubes* is defined as

$$D^{(d)} = \bigcup_{k=0}^{\infty} D_k^{(d)}.$$

THEOREM 8.2. *If S_1 and S_2 are discrete subsets of \mathbf{R}^d and*

$$(1) \quad ||S_r \cap Q| - \alpha \lambda_d(Q)| \leq C \cdot s(Q)^{d-1-\varepsilon}$$

holds for every dyadic cube Q and $r = 1, 2$, then there is a bijection ϕ from S_1 onto S_2 such that $|\phi(x) - x| \leq M$ for every $x \in S_1$, where the constant M only depends on d, C, ε and α .

Proof of 8.1. We put $\alpha = \lambda_k(H_r)$ ($r = 1, 2$). If $(a_1, \dots, a_d) \in \mathbf{R}^d$ then we shall abbreviate the linear combination $a_1 x_1 + \dots + a_d x_d$ by $a \cdot x$. Similarly, if $(b_1, \dots, b_k) \in \mathbf{R}^k$ then $b \cdot e$ will denote the linear combination $b_1 e_1 + \dots + b_k e_k$.

Suppose that x_1, \dots, x_d satisfy conditions (i) and (ii) of 8.1. Let $x = (x_1, \dots, x_d)$, $n = (n_1, \dots, n_d)$ and put

$$S_r(u) = \{n \in \mathbf{Z}^d : (u + n \cdot x) \in H_r\} \quad (u \in \mathbf{R}^k, r = 1, 2).$$

We shall prove that (1) holds for every lattice cube Q , $S_r = S_r(u)$ and $r = 1, 2$.

Let $Q = \prod_{i=1}^d [a_i, a_i + N)$. Then putting $a = (a_1, \dots, a_d)$ and $m = (m_1, \dots, m_d)$, we have

$$\begin{aligned} |S_r(u) \cap Q| &= \\ |\{n \in \mathbb{Z}^d : a_i \leq n_i < a_i + N \ (i = 1, \dots, d), (u + n \cdot x) \in H_r\}| &= \\ |\{m \in \mathbb{Z}^d : 0 \leq m_i < N \ (i = 1, \dots, d), (u + a \cdot x + m \cdot x) \in H_r\}| &= \\ |F_N(u + a \cdot x; x_1, \dots, x_d) \cap H_r|. \end{aligned}$$

Consequently,

$$\left| \frac{1}{N^d} |S_r(u) \cap Q| - \alpha \right| = D(F_N(u + a \cdot x; x_1, \dots, x_d); H_r)$$

and hence (1) follows from condition (ii) and from $\lambda_d(Q) = N^d$.

Thus, by 8.2, there is bijection ϕ_u from $S_1(u)$ onto $S_2(u)$ such that

$$(2) \quad |\phi_u(z) - z| \leq M \text{ for every } z \in S_1(u)$$

where the constant M only depends on d, C, ε and α . The important point here is that M does not depend on u .

Let G denote the additive group generated by x_1, \dots, x_d and the unit vectors $e_1 = (0, \dots, 0, 1, 0, \dots, 0)$ ($i = 1, \dots, k$). We define the equivalence relation \sim by

$$z_1 \sim z_2 \Leftrightarrow z_1 - z_2 \in G \quad (z_1, z_2 \in \mathbb{R}^k).$$

Let E be an equivalence class and let an element $u \in E$ be selected. Then every element $z \in E$ has a unique representation of the form

$$z = u + n \cdot x + m \cdot e \quad (n \in \mathbb{Z}^d, m \in \mathbb{Z}^k).$$

If $z \in H_1$, then $(u + n \cdot x) \in H_1$ and hence $n \in S_1(u)$. Let $\phi_u(n) = n'$. As $n' \in S_2(u)$, we have $(u + n' \cdot x) \in H_2$ and hence there is $m' \in \mathbb{Z}^k$ such that $u + n' \cdot x + m' \cdot e \in H_2$. Let $\chi_u(z) \stackrel{\text{def}}{=} u + n' \cdot e$. Then χ_u is a well-defined map from $H_1 \cap E$ to $H_2 \cap E$. (Note that m' is uniquely determined). As ϕ_u is a bijection from $S_1(u)$ onto $S_2(u)$, χ_u is a bijection from $H_1 \cap E$ onto $H_2 \cap E$.

By (2), $|n' - n| \leq M$. Since $z, \chi_u(z) \in I^k$, we have $|(n' - n) \cdot x + (m' - m) \cdot e| = |\chi_u(z) - z| \leq \sqrt{k}$. This implies

$$|m' - m| \leq |(m' - m) \cdot e| \leq \sqrt{k} + |(n' - n) \cdot x| \leq \sqrt{k} + M \cdot \max_i |x_i|.$$

We have proved that for every $z \in H_1$ there are vectors $a \in \mathbb{Z}^d$ and $b \in \mathbb{Z}^k$ such that

$$(3) \quad |a| \leq M, |b| \leq \sqrt{k} + M \cdot \max_i |x_i|,$$

and $\chi_u(z) = z + a \cdot x + b \cdot e$.

Let $\{d_t\}_{t=1}^K$ be an enumeration of the vectors $a \cdot x + b \cdot e$, where $a \in \mathbb{Z}^d$ and $b \in \mathbb{Z}^k$ satisfy (3). We have proved that for every $z \in H_1 \cap E$ there is $1 \leq t \leq K$ such that $\chi_u(z) = z + d_t$. Since the equivalence class E was selected arbitrarily and $d_t \in G$ for every t , this implies that there is a bijection χ from H_1 onto H_2 such that for every z there is a t with $\chi(z) = z + d_t$. Let

$$A_t = \{z \in H_1 : \chi(z) = z + d_t\} \quad (t = 1, \dots, K).$$

Then $\cup_{t=1}^K A_t$ and $\cup_{t=1}^K (A_t + d_t)$ are disjoint decompositions of H_1 and H_2 , respectively, and this completes the proof of the theorem.

9. Two applications: Cavalieri's principle and circle-squaring.

We shall need the following discrepancy estimate.

LEMMA 9.1. *For almost every $x_1, \dots, x_d \in I^k$ and for every $\varepsilon > 0$ there is a constant $C > 0$ such that*

$$D(F_N(u; x_1, \dots, x_d)) \leq C \cdot (\log N)^{k+d+\varepsilon} \cdot N^{-d}$$

holds for every $u \in \mathbb{R}^k$ and $N = 2, 3, \dots$

This lemma is an easy consequence of the Erdős–Turán–Koksma formula ([11], p. 116) and an estimate given by W. Schmidt on sums of fractional parts of some special sequences ([24], p. 517). For the details, see [13] Section 8 and [16] Lemma 4.

Let A be a measurable subset of \mathbf{R}^2 . By Fubini's theorem, $\lambda_2(A) = \int \lambda_1(A^y) dy$, where $A^y = \{x : (x, y) \in A\}$. This implies that if A_1 and A_2 are both measurable and $\lambda_1(A_1^y) = \lambda_1(A_2^y)$ holds for every y then $\lambda_2(A_1) = \lambda_2(A_2)$. (This statement is sometimes referred to as Cavalieri's principle.)

As an application of the one-dimensional variant of 8.1 we shall prove that under suitable assumptions on the sections A_1^y, A_2^y , the condition $\lambda_1(A_1^y) = \lambda_1(A_2^y)$ ($y \in \mathbf{R}$) implies that A_1 and A_2 are equidecomposable. The measurability of the sets A_1 and A_2 is not required.

THEOREM 9.2. *Let A_1, A_2 be bounded subsets of \mathbf{R}^2 . Suppose that there are positive numbers k and δ such that, for every $y \in \mathbf{R}$,*

- (i) *the sections A_1^y and A_2^y consist of at most k intervals, and*
- (ii) *either $A_1^y = A_2^y = \emptyset$ or $\lambda_1(A_1^y) = \lambda_1(A_2^y) > \delta$.*

Then A_1 and A_2 are equidecomposable using translations.

Proof. By 9.1 we can select real numbers x, y such that $D(F_N(u; x, y)) \leq C \cdot \log^4 N / N^2$ for every $u \in \mathbf{R}$. By (i), this implies that $D(F_N(u; x, y); A_i^y) \leq k \cdot C \cdot \log^4 N / N^2$ for every $u \in \mathbf{R}$ and $i = 1, 2$. According to 8.1, this implies that A_1^y and A_2^y are equidecomposable using translations by the numbers $n_1 x + n_2 y$ ($|n_1|, |n_2| \leq M$), where M only depends on $\lambda(A_i^y)$. Since this is bounded from below by (ii), M does not depend on y . This implies that A_1 and A_2 are equidecomposable, using the translations by the vectors $(n_1 x + n_2 y, 0)$ ($|n_1|, |n_2| \leq M$).

THEOREM 9.3. *If $A_1, A_2 \subset \mathbf{R}^k$ are bounded convex sets with $\lambda_k(A_1) = \lambda_k(A_2) > 0$, then $A_1 \stackrel{T_k}{\sim} A_2$.*

Proof. We may assume that $A_1, A_2 \subset I^k$. If $F \subset I^k$ is a finite set then we have $D(A_i; F) \leq C \cdot D(F)^{1/k}$ ($i = 1, 2$), where the constant C only depends on k (see [11], Theorem 1.6, p. 95). Let $d > k$, then an application of 8.1 and 9.1 proves the theorem.

A similar argument shows that the convexity of the sets can be replaced by the property that the box dimension of the boundary of the sets is less than k . For $k = 2$ this condition is satisfied, for example, if the sets

are Jordan domains with rectifiable boundary (see [16]).

If $H \subset [0, 1]$ then the box dimension $\Delta(H)$ of H is the infimum of the numbers α such that

$$|\{i : 1 \leq i \leq n, H \cap \left[\frac{i-1}{n}, \frac{i}{n}\right) \neq \emptyset\}| \leq n^\alpha$$

if n is large enough. It follows from [16] Theorem 3, that if $H \subset [0, 1]$ and $\Delta(\partial H) < 1$ then H is equidecomposable to an interval. C.A. Rogers asked whether or not the set

$$A = \left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{7}{8}, \frac{8}{9}\right) \cup \left(\frac{25}{27}, \frac{26}{27}\right) \cup \dots$$

is equidecomposable to $(0, 1/2)$ ([30] pp. 119 and 230). It is easy to check that $\Delta(\partial A) = 0$ and hence, by the result quoted above, the answer to Roger's question is affirmative.

The set

$$B = \bigcup_{n=1}^{\infty} \left(\frac{1}{2n}, \frac{1}{2n-1}\right)$$

is also equidecomposable to an interval, since $\Delta(\partial B) = 1/2$. We do not know, however, whether or not *every* union of a convergent sequence of intervals is equidecomposable to an interval. *

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(*) Added in proof: Recently I gave a negative answer; see my paper "Decomposition of sets of small or large boundary" to appear in *Mathematika*.

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