TWO APPLICATIONS OF SINGULAR SETS TO THE THEORY OF COMPACTIFICATIONS (*)

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SOMMARIO. - *In questo lavoro si usa una costruzione, che generalizza quella delle compattificazioni singolari, per provare due risultati di teoria delle compattificazioni. Questi due risultati non sono tra loro correlati, tranne per il fatto che illustrano l'uso di questa costruzione. Questa tecnica, che è stata usata implicitamente in molti recenti lavori sulle compattificazioni, viene qui resa esplicita.*

SUMMARY. - *In this paper we use a construction, which generalizes the construction of a singular compactification, to prove two results from the theory of compactifications. These results are unrelated except that they illustrate the use of this construction. This technique has been implicit in much of recent work in compactification theory and is made explicit here.*

Introduction. In [10], G.T. Whyburn introduced the concept of the *singular* set of a continuous function and used the notion for the study of compactifications of mappings. Since that time the idea has been implicitly used, in a form which often seemed to be unrelated to the original definition, by many authors to study compactifications of locally compact spaces. A few of these are [1,2,4,5, 8,9]. In this paper we define a space, which is equivalent to Whyburn's *unified space* and use this to investigate two areas of compactification theory.

The first of these is as follows: with each compactification αX of a Tychonoff space X we can associate the Banach algebra $C_\alpha(X)$, consist-

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ing of all real-valued continuous functions on X which have continuous extensions to αX. $C_\alpha(X)$ completely determines αX. It is, therefore, of interest to study algebraic and topological properties of $C_\alpha(X)$.

In section 2 of this paper we show that, if X is locally compact and αX has 0-dimensional remainder, then certain collections of closed continuous functions form a set of generators for $C_\alpha(X)$. In particular, if X is also paracompact and ϕX is the Freudenthal compactification of X, then $C_\phi(X)$ is the closed subalgebra of $C^*(X)$ generated by the collection of all closed functions.

For the second application, we consider a lattice problem of the singular compactifications of the natural numbers (and, more generally, of any discrete space). The collection of the singular compactifications of a locally compact space, that is the compactifications whose remainders are retracts (see [4,5,8]), is always a complete lower semilattice [8]. On the other hand, it is rarely a complete lattice. In fact it is known that, if X is not pseudocompact, then βX is not singular [7] but is *sup* of singular compactifications [4]. This can happen also for pseudocompact spaces [3, ex. 2]. One of the unsolved problems in the theory is: when is the collection of singular compactifications a lattice? In [4] the following theorem is claimed:

THEOREM. *If the set of singular compactifications forms a lattice, then it forms a complete lattice.*

The proof of this theorem in [4] is incorrect. The truth or falsity of the claim remains unknown. One of the consequences of it would be that the singular compactifications of the natural numbers do not form a lattice. In section 3 we prove that this remains true independent of the above theorem. Notice that, when βX is singular, every compactification of X is singular [8], so the singular compactifications of X trivially form a complete lattice. The principal unresolved conjecture in this theory is the following:

CONJECTURE. *The singular compactifications of a space X form a lattice if and only if βX is singular.*

This would imply that the above theorem is true and that it is a rare occurrence for the singular compactifications to form a lattice.

1. All spaces considered in this paper are Hausdorff and locally compact. Let X and Y be spaces and let f be a continuous map from X to Y. We recall that the *singular set* of f is the set

$$\mathcal{S}(f) = \{y \in Y \mid \forall U \in \mathcal{N}_Y(y), \overline{f^{-1}(U)} \text{ is not compact}\},$$

where $\mathcal{N}_Y(y)$ is the set of the open neighbourhoods of y in Y. It is easy to see that $\mathcal{S}(f)$ is a closed subset of Y and $\mathcal{S}(f) \subset \overline{f(X)}$. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the topologies of X and Y respectively. One can easily check that the family

$$\mathcal{T}_X \cup \{U \cup (f^{-1}(U) \setminus F) \mid U \in \mathcal{T}_Y,\ F \subset X,\ F \text{ compact}\}$$

is a base for a topology $\mathcal{T}$ on $X \cup Y$. We denote the space $(X \cup Y, \mathcal{T})$ by the symbol $S(X,Y,f)$. One can easily show that $S(X,Y,f)$ is Hausdorff and that the original topologies are induced on the subspaces X and Y. Clearly Y is a closed subset and $Cl_{S(X,Y,f)}(X) = X \cup \mathcal{S}(f)$. We note that, if K is a compact neighbourhood of a point $y \in Y$, then $K \cup f^{-1}(K)$ is a compact neighbourhood of y in $S(X,Y,f)$. So $S(X,Y,f)$ is locally compact and it is compact if and only if Y is. Let $r : S(X,Y,f) \to Y$ defined by $r(z) = f(z)$ if $z \in X$ and $r(z) = z$ if $z \in Y$. It is easy to see that r is continuous, hence it is a retraction.

In a method analogous to that used in [10] one can prove that the space $S(X,Y,f)$ is homeomorphic to the *unified space* defined by Whyburn.

If Y is compact, and X is not, then $X \cup \mathcal{S}(f) = Cl_{S(X,Y,f)}(X)$ is a compactification of X. We will denote it by $\omega_f X$. Note that every compactification αX of X can be obtained in this way. In fact, if j is the inclusion map of X into αX, then $\omega_j X$ is equivalent to αX.

PROPOSITION 1.1. *Let f be a continuous map from X to a compact space Y. Then $\omega_f X$ is the minimal compactification of X to which f extends.*

Proof. The retraction r defined above is an extension of f to all of $S(X,Y,f)$, in particular to $\omega_f X$. Moreover that extension is 1-1 on the remainder. The conclusion follows from [6].

PROPOSITION 1.2. *Let $\mathcal{F} = \{f_i : X \to K_i\}$, where, for each i, f_i is continuous and K_i is a compact space. Let $h : X \to \prod K_i$, be the evaluation map. Then $\omega_h X$ is the minimal compactification to which every f_i extends, that is $Sup\{\omega_{f_i} X\} = X \cup S(h)$.*

Proof. It is clear that h extends to a compactification αX of X if and only if each f_i extends to αX. Thus the conclusion follows by Prop. 1.1.

2. In this section we investigate the algebra of real-valued continuous functions which extend to a compactification with 0-dimensional remainder. We need some preliminary results.

LEMMA 2.1. *Let X be a σ-compact space and let $f : X \to Y$ be continuous. Then one has:*

(i) if Y is first countable then $S(X,Y,f)$ is first countable at each point of Y.

(ii) if Y is compact, $y \in S(f)$ and $S(f)$ is first countable at y, then $Cl_{S(X,Y,f)}(X) = X \cup S(f)$ is first countable at y.

Proof. By hypothesis, one has $X = \cup_{h \in N} F_h$, with F_h compact and $F_h \subset Int(F_{h+1})$. Thus every compact $F \subset X$ is contained in F_h, for a suitable h. To prove *(i)*, let $y \in Y$. If $\{U_h\}_{h \in N}$ is a (decreasing) local basis of y in Y, then it is easy to see that $\{U_h \cup (f^{-1}(U_h) \setminus F_h)\}_{h \in N}$ is a (decreasing) local basis at y in $S(X,Y,f)$.

The hypotheses of *(ii)* imply that $K = X \cup S(f)$ is compact and there exists a countable family $\{V_h\}$ of open subsets of Y such that $\{V_h \cap S(f)\}$ is a local basis for y in $S(f)$. Then $\{(V_h \cup (f^{-1}(V_h) \setminus F_h)) \cap K\}_{h \in N}$ is a family of open neighbourhoods of y in K whose intersection is y. Choose, for each h, a closed neighbourhood T_h of y in K, which is contained in $(V_h \cup (f^{-1}(V_h) \setminus F_h)) \cap K$. Since $\cap_{h \in N} T_h = \{y\}$, an easy compactness argument shows that $\{T_h\}$ is a local basis at y in K.

Since every compactification αX of X is equivalent to $X \cup S(f)$ for a suitable f, the *(ii)* part of the above lemma yields the following

THEOREM 2.2. *Suppose X is (locally compact and) σ-compact. Let*

αX be a compactification of X which has a first countable remainder. Then αX is first countable at each point of $\alpha X \backslash X$.

LEMMA 2.3. *Let X be a paracompact space, Y a first-countable space and $f : X \to Y$ a closed continuous map. Then $S(f)$ is a discrete subspace of Y.*

Proof. It is known that a locally compact paracompact space is a topological sum of σ-compact spaces. First we will prove the statement in the case X σ-compact. Let y be a limit point of $S(f)$. Let $\{U_k\}_{k \in N}$ be a local basis of y in Y, with $U_k \subset U_{k+1}$. By Lemma 2.1 *(i)*, y has a decreasing countable local basis, in $S(X, Y, f)$, of the form $\{U_k \cup (f^{-1}(U_k) \backslash F_k)\}_{k \in N}$, where F_k's are suitable compact subsets of X. For each k, the set $V_k = U_k \backslash \{y\}$ is an open subset of Y which meets $S(f)$. Then one has $f^{-1}(V_k) \backslash F_k \neq \emptyset$. Let x_k be a point of $f^{-1}(V_k) \backslash F_k$, for each k. One has $x_k \in U_k \cup (f^{-1}(U_k) \backslash F_k)$, so the sequence $\{x_k\}$ converges to y in $S(X, Y, f)$. Thus, $\{x_k | k \in \mathrm{N}\}$ is a closed subset of X and this implies that $A = \{f(x_k) | k \in \mathrm{N}\}$ is a closed subset of Y. Moreover $y \notin A$. Let r be the retraction $S(X, Y, f)$ onto Y defined in section 1. One has $r(x_k) \to y$. But $r(x_k) = f(x_k)$, hence $y \in Cl_Y(A) = A$, contradition.

Before we start the proof of the general case, we need to prove the following statement, for which we need only the hypothesis that f is a closed continuous function: if $y \in S(f)$ and $f^{-1}(y) \subset Z$, where Z is an open subset of X, then $y \in S(f|_Z)$. In fact, suppose that there exists $V \in \mathcal{N}_Y(y)$ such that $(f|_Z)^{-1}(V) = f^{-1}(V) \cap Z$ is relatively compact (in Z, hence in X). Put $E = Cl_X f^{-1}(V) \cap (X \backslash Z)$. E does not meet $f^{-1}(y)$. Then, since f is closed, $V_1 = V \backslash f(E)$ is an open subset of Y containing y. One has $f^{-1}(V_1) \subset f^{-1}(V) \cap Z$ and this implies that $f^{-1}(V)$ is relatively compact, contradiction.

Now, let X be the topological sum of a family $\{X_\lambda\}$ of σ-compact spaces. Again suppose y is a limit point of $S(f)$. We observe that $S(f) \subset f(X)$, because f is closed. Let $\{y_n\}$ be a sequence in $S(f) \backslash \{y\}$ which converges to y and let $T = \{y_n | n \in \mathrm{N}\}$. Then $f^{-1}(T)$ meets infinitely many X_λ's. Suppose not, that is $f^{-1}(T) \subset Z$, where Z is the sum of a finite subfamily of $\{X_\lambda\}$. Then we have $T \subset S(f|_Z)$ and this implies that y is a limit point of $S(f|_Z)$. But Z is also σ-compact and $f|_Z$ is a closed map, so we have a contradiction. Obviously, for any subsequence of $\{y_n\}$, the inverse image meets infinitely many X_λ's. Now we will define a sequence $\{x_k\}$ in $f^{-1}(T)$, inductively. We take $x_1 \in f^{-1}(y_1)$. Suppose we have defined $x_1, \ldots, x_{k-1}$, with $x_i \in X_{\lambda_i}$. Then the inverse image of $T' = T \backslash \{f(x_i) | i = 1, \ldots, k-1\}$ cannot be contained in $\bigcup_{i=1}^{k-1} X_{\lambda_i}$, so we choose $x_k \in f^{-1}(T')$, $x_k \in X_{\lambda_k}$ with $\lambda_k \neq \lambda_1, \ldots, \lambda_{k-1}$. Obviously, $G = \{x_k | k \in N\}$ is a closed subset of X, so $f(G)$ is a closed subset of Y. But $f(G)$ is an infinite subset of T,hence y is a limit point of $f(G)$ and $y \notin f(G)$, contradiction.

Let αX be a compactification of X. We denote by $C_\alpha(X)$ the algebra of real-valued continuous functions on X which extend to αX and by $\mathcal{F}_\alpha$ the family of the closed functions which belong to $C_\alpha(X)$.

LEMMA 2.4. *Let αX be a compactification of a (locally compact) space X, with 0-dimensional remainder. Then $C_\alpha(X)$ is the closure, with respect to the uniform convergence topology, of the algebra generated by $\mathcal{F}_\alpha$.*

Proof. It is well known that the "extension" map is a canonical isomorphism between $C_\alpha(X)$ and $C(\alpha X)$, which is also an homeomorphism with respect to the u.c. topology. Let $\mathcal{G}$ be the set of the extensions to αX of the elements of $\mathcal{F}_\alpha$. In view of Stone-Weierstrass theorem, we need only to prove that $\mathcal{G}$ separates points of αX. Let $y, z \in \alpha X$. First suppose $y \in X$. We choose $U \in \mathcal{N}_X(y)$ relatively compact and such that $z \notin U$. Let $g \in C(\alpha X)$ be such that $g(y) = 1$ and $g(\alpha X \backslash U) = 0$. It is easy to see that a continuous map which takes only a finite numbers of values out of a compact set is a closed map. So the restriction of g to X is closed. Thus $g \in \mathcal{G}$ and it separates y and z. Now, let $y, z \in K = \alpha X \backslash X$. We can find an open and closed subset A of K such that $y \in A$ and $z \in B = K \backslash A$. Since A and B are disjoint closed subsets of αX, there exist open subsets V and W of αX containing A and B, respectively, and such that $Cl_{\alpha X}(V) \cap Cl_{\alpha X}(W) = \emptyset$. So we can find $h \in C(\alpha X)$ such that $h(V) = 1$ and $h(W) = 0$. The restriction of h to X is a closed map because it takes only two distinct values out of the compact set $X \backslash (V \cup W)$. So $h \in \mathcal{G}$ and it separates y and z.

We denote by ϕX the Freudenthal compactification of X and by $\mathcal{F}$ the set of bounded closed continuous real-valued functions defined on X.

THEOREM 2.5. *If X is a paracompact space, then $C_\phi(X)$ is the closure, with respect to the uniform convergence topology, of the algebra generated by $\mathcal{F}$.*

Proof. In view of Lemma 2.4, we need only to prove $\mathcal{F} \subset C_\phi(X)$. Let $f \in \mathcal{F}$. By Prop 1.1 and Lemma 2.3, f extends to a compactification with discrete (hence finite) remainder. This implies that f extends to ϕX.

EXAMPLE. Let $X = [0, \omega_1[\times I$, where $[0, \omega_1[$ is the space of countable ordinals, and I is the unit closed real interval. Let $p : X \to I$ be the projection. Since the ordinal space is countably compact (and I is first-countable), p is a closed map. We note that $S(p) = I$ is not discrete. It is known that $\beta X = [0, \omega_1] \times I$ and it is easy to check that $S(X, I, p) = \omega_p X = \beta X$, that is βX is the only compactification to which p extends. In particular, p does not extend to ϕX (since $\beta X \backslash X$ is connected, ϕX is the one-point compactification of X).

3. Let f be a continuous map from X to a compact space K. Suppose f is a *singular map*, that is $S(f) = K$. In this case $\omega_f X = S(X, K, f)$ is a compactification of X having the remainder as a retract. When this is true, $\omega_f X$ is denoted by the symbol $X \cup_f K$ and it is said to be a *singular compactification*. Conversely, if a compactification αX is such that $\alpha X \backslash X$ is a retract of αX, then there exists a singular map $f : X \to \alpha X \backslash X$ such that αX is equivalent to $X \cup_f (\alpha X \backslash X)$. Properties of singular compactifications can be found in [2,4,5,8]. Let $g : X \to K$ be any continuous map (K compact). We will now consider several conditions which will imply that $\omega_g X$ is singular.

PROPOSITION 3.1. *Let g be a continuous map from X to a compact space K. Then $\omega_g X$ is a singular compactification if and only if there exists a continuous map $f : X \to S(g)$ such that, for each $y \in S(g)$ and for each $U \in \mathcal{N}_K(y)$, there is a $V \in \mathcal{N}_K(y)$ and a compact $F \subset X$ such that $g^{-1}(V) \backslash F \subset f^{-1}(U \cap S(g))$.*

Proof. Let $\omega_g X$ be singular. Then there is a singular map $f : X \to \mathcal{S}(g)$ such that $\omega_g X$ is equivalent to $X \cup_f \mathcal{S}(f) = X \cup_f \mathcal{S}(g)$. Let $y \in \mathcal{S}(g)$ and let $U \in \mathcal{N}_K(y)$. Put $W = U \cap \mathcal{S}(g)$. The set $W \cup f^{-1}(W)$ is a basic open neighbourhood of y in $X \cup_f \mathcal{S}(g)$. Then it must contain a basic neighbourhood of y with respect to the topology of $\omega_g X$. So there exists $V \in \mathcal{N}_K(y)$ and a compact $F \subset X$ such that $\omega_g X \cap (V \cup (g^{-1}(V) \setminus F)) \subset W \cup f^{-1}(W)$. The conclusion easily follows. Conversely, the hypothesis implies that f is singular. Let $y \in \mathcal{S}(g)$ and let $W \cup (f^{-1}(W) \setminus G)$ a basic neighbourhood of y in $X \cup_f \mathcal{S}(g)$. Let $U \in \mathcal{N}_Y(y)$ be such that $W = U \cap \mathcal{S}(g)$ and let V, K be as in the hypothesis. We can choose $V \subset U$, so $V \cap \mathcal{S}(g) \subset W$. Thus $y \in \omega_g X \cap (V \cup (g^{-1}(V) \setminus (F \cup G)) \subset W \cup (f^{-1}(W) \setminus G)$. This proves that $\omega_g X$ is equivalent to the singular compactification $X \cup_f \mathcal{S}(g)$.

PROPOSITION 3.2. *Let g be a continuous map from X to a compact space K. If $\mathcal{S}(g)$ is a retract of $\overline{g(X)}$ then $\omega_g X$ is a singular compactification.*

Proof. Let r be the natural retraction of $\mathcal{S}(X, K, g)$ onto K. Since $r(X) = g(X)$, the restriction r_1 of r to $X \cup \overline{g(X)}$ can be considered a retraction from $X \cup \overline{g(X)}$ to $\overline{g(X)}$. By hypothesis, there exists a retraction $s : \overline{g(X)} \to \mathcal{S}(g)$. Thus the restriction of $s \circ r_1$ to $X \cup \mathcal{S}(g) = \omega_g X$ is a retraction of $\omega_g X$ onto its remainder $\mathcal{S}(g)$.

The next example shows that the condition in the above proposition is not necessary for $\omega_g X$ to be singular

EXAMPLE 1. Let $X =]-1, 0] \cup [1, 2[\subset \mathbf{R}$ and let $K = [-1, 1]$. Let $g : X \to K$ be defined by $g(x) = x$ for $x \in]-1, 0]$ and $g(x) = x - 1$ for $x \in [1, 2[$. Clearly, $\mathcal{S}(g) = \{-1, 1\}$, so $\omega_g X$ in the (unique) 2-point compactification of X. Then it is easy to see that $\omega_g X$ is singular. However, $\mathcal{S}(g)$ is not a retract of $\overline{g(X)} = K$.

Now we use the above results to investigate when the *supremum* of a family of singular compactifications is also singular. From Prop. 3.2 and 1.2 we easily obtain

COROLLARY 3.3. *Let $\{X \cup_{f_i} K_i\}$ be a family of singular compactifications of X. Let $h : X \to \prod K_i$ be the diagonal map. If there exists a retraction from $\overline{h(X)}$ to $S(h)$, then $Sup\{X \cup_{f_i} K_i\}$ is a singular compactification.*

COROLLARY 3.4. *Let $X \cup_f K$ and $X \cup_g T$ be singular compactifications with finite remainders. Then $Sup\{X \cup_f K, X \cup_g T\}$ is also singular.*

Proof. Let $h = f \times g$. Since $K \times T$ is a discrete space, obviously there exists a retraction from (the closure of) $h(X)$ to $calS(h)$.

Let $\{X \cup_{f_i} K_i\}$ be an arbitrary family of singular compactifications of X. One can deduce from Prop 1.2 that, whenever the diagonal map h is singular, $Sup\{X \cup_{f_i} K_i\}$ is a singular compactification. In fact it is sufficient the condition $S(h) = \overline{h(X)}$. The following example shows that this condition is not necessary.

EXAMPLE 2. Let $\mathbf{N} = \cup_{i=1}^{4} M_i$ be a partition of $\mathbf{N}$ such that M_1 is finite and M_i is infinite for $i \neq 1$. Put $A_1 = M_1 \cup M_2$, $A_2 = M_3 \cup M_4$, $B_1 = M_1 \cup M_3$, $B_2 = M_2 \cup M_4$. Let $K = \{1,2\}$ and let $f, g : \mathbf{N} \to K$ be defined by $f(A_i) = i, g(B_i) = i, i = 1,2$. Clearly f and g are singular maps. By Cor. 2.4, $Sup\{X \cup_f K, X \cup_g K\}$ is a singular compactification (in fact this can be checked directly, because every compactification of $\mathbf{N}$ with finite remainder is singular). Clearly $h = f \times g$ is not singular because $h^{-1}\{(1,1)\} = A_1 \cap B_1 = M_1$ is compact. Since h is onto, $S(h)$ is different from (the closure of) $h(X)$.

The above example shows that Thm. 8 of Ch. III in [8] is incorrect. This result was used in the proof of the theorem in [4] which was mentioned in the Introduction.

By Prop. 1.2 and 3.1 we have

THEOREM 3.5. *Let $\{X \cup_{f_i} K_i\}$ be a family of singular compactifications of X. Let $h : X \to \prod K_i$ be the diagonal map. Then $Sup\{X \cup_{f_i} K_i\}$ is a singular compactification if and only if there exists $m : X \to S(h)$ such that, for each $y \in S(h)$ and for each (basic) neighbourhood U of y*

in $\prod K_i$, there is a (basic) neighbourhood V of y in $\prod K_i$, and a compact $F \subset X$ such that $h^{-1}(V)\backslash F \subset m^{-1}(U \cap S(h))$.

In order to conclude, we will need the following

LEMMA 3.6. *Let $\{X \cup_{f_i} K_i\}$ be a family of singular compactifications of X. Let $h : X \to \prod K_i$ be the diagonal map. Suppose that every point of $h(X)$ is an isolated point of $\overline{h(X)}$. Then $Sup\{X \cup_{f_i} K_i\}$ is a singular compactification if and only if $S(h)$ is a retract of $\overline{h(X)}$.*

Proof. In view of Prop. 3.2, we need only to prove one implication. Let $Sup\{X \cup_{f_i} K_i\}$ be a singular compactification and let m be as in Thm. 3.5. Let $s : h(X) \to X$ be such that $s \circ h = 1_{h(X)}$. It is known ([2], Lemma 1.2) that $\overline{h(X)}\backslash h(X) \subset S(h)$, hence $\overline{h(X)} = h(X) \cup S(h)$. Thus we can define $r : \overline{h(X)} \to S(h)$ by putting $r(y) = y$ for $y \in S(h)$ and $r(y) = m(s(y))$ for $y \notin S(h)$. We need only to prove that r is continuous. Clearly r is continuous in the points of $h(X)$, because they are isolated points. Now, let $y \notin h(X)$ and let U be a basic neighbourhood of y in $\prod K_i$. One has $y \in S(h)$, hence there is a (basic) neighbourhood V of y in $\prod K_i$, and a compact $F \subset X$ such that $h^{-1}(V)\backslash F \subset m^{-1}(U \cap S(h))$ (see Thm. 3.5). We can choose $V \subset U$. Put $W = V\backslash h(F)$. Since $y \notin h(X)$, W is also a neighbourhood of y in $\prod K_i$. We want to show that, for each $z \in W \cap h(X)$, one has $r(z) \in U$. If $z \in S(h)$, this is trivially true. Otherwise, one has $z \in h(X)$. Put $x = s(z)$. One has $x \in h^{-1}(W) \subset h^{-1}(V)\backslash F \subset m^{-1}(U \cap S(h))$. Therefore one has $r(z) = m(s(z)) = m(z) \in U$.

And now,

THEOREM 3.7. *Let X be an infinite discrete space. Then the set of the singular compactifications of X is not a lattice.*

Proof. Let t be a bijective map from X to $X \times X$ and let $p_1, p_2 : X \times X \to X$ be the projections. Put $f' = p_1 \circ t$, $g' = p_2 \circ t$. Let ωX be the one-point compactification of X, and let j_1, j_2 be the inclusion maps of X into ωX and βX, respectively. If we set $f = j_1 \circ f' : X \to \omega X, g =$

$j_2 \circ g' : X \to \beta X$, then f and g are singular maps. We want to show that $Sup\{X \cup_f \omega X, X \cup_g \beta X\}$ is not a singular compactification. Put $h = f \times g$. We claim that $S(h) = (\omega X \times \beta X)\backslash(X \times X)$. To see that, let $y = (x_1, x_2) \in X \times X$. Then $h^{-1}(y) = f^{-1}(x_1) \cap g^{-1}(x_2)$ is a singleton. Since y is an isolated point, one has $y \notin S(h)$. We note that $h(X) = X \times X$ is dense in $\omega X \times \beta X$. Thus, if $y \in (\omega X \times \beta X)\backslash(X \times X)$, then $y \in S(h)$ ([2], lemma 1.2). Suppose that $Sup\{X \cup_f \omega X, X \cup_g \beta X\}$, is a singular compactification. Then, by Lemma 3.6, there exists a retraction $r : \omega X \times \beta X \to (\omega X \times \beta X)\backslash(X \times X)$. Let $x_0 \in X$ and $E = \{x_0\} \times \times \beta X$. E is (homeomorphic to βX and it is) an open and closed subset of $\omega X \times \beta X$. Thus, $r|_E$ must map all points of E, except a finite number, into $E\backslash h(X) = \{x_0\} \times (\beta X \backslash X)$. We can slightly modify $r|_E$ to obtain a retraction from E to $E\backslash h(X)$, that is, up the homeomorphism, a retraction of βX onto $\beta X \backslash X$. This is impossible, by [7].

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