ON TOPOLOGY AND DIMENSIONS OF RECURRENT UNIFORM CANTOR SETS (*)

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SOMMARIO. - *Su un insieme di Cantor ricorrente, verificante una condizione di uniformità, si introduce una metrica naturale che consente di stimare la sua dimensione di Hausdorff, provando inoltre che essa coincide con la sua dimensione box-counting, generalizzando a spazi metrici ben noti teoremi validi per insiemi autosimili.*

SUMMARY. - *Introducing a natural metric on a recurrent uniform Cantor set we are able to estimate its Hausdorff dimension and show that it coincides with its box-counting dimension, generalizing in a metric space well-known theorems about self-similar sets.*

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1. Introduction. The most common way of introducing fractal sets is to consider such objects as limits of a recursively defined geometric construction. This classic approach is widely exploited in several examples by Mandelbrot [10].

The basic model of such constructions is given by the ternary Cantor set. A particular feature which is involved in the construction of these sets is that at each step we cut off an interval of the same diameter as the two which are left. In this manner the limit set inherits some properties

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of metric regularity; in particular the Hausdorff dimension and the box-counting dimension have the same value.

The equality of the dimensions is shared by a braod class of fractals, indeed it was present in the earlier examples of these sets. Actually such a situation was tacitly assumed as a general property. See Kaplan and Yorke [9]. Subsequently, some sets were produced, for which the Hausdorff dimension is strictly less than the box-counding dimension. Elementary but significant examples can be found in Bedford [4] and McMullen [11].

The Hausdorff measure generalizes, in the euclidean spaces, the Lebesgue measure and if the space is n-dimensional, we can find subsets of any dimension α, with $0 \leq \alpha \leq n$. Hence, it is preferred for its theoretical meaning, but the estimate in general, is difficult. For this reason, sometimes in concrete examples, the box-counting dimension is used, since its calculation is easier to perform.

In [3], Beardon calls *general Cantor sets* the sets obtained through a process that generalizes in a complete metric space the Cantor-type procedure.

In this note we consider as in [3], the geometric construction employed in the definition of the ternary Cantor set. Such constructions are carried out through a family of compact subsets, called *geometric scheme* and they may be of the following two types: schemes of Cantor type and of Markov type.

Schemes of Cantor type are termed the geomtric schemes in which at any n-th step of the recursive definition, N^n subsets of the scheme are involved, being N a positive integer fixed in advance; while, the schemes formed by a subfamily of sets of a Cantor type scheme in which the sets are selected, according to a transition matrix, are said to be of Markov type. These latter lead to the construction of recurrent sets which generalize those of Cantor-type. In particular we extend in the recurrent case, the method of estimating the Hausdorff dimension presented in [3].

Geometric schemes satisfying a suitable uniformity condition are called *uniform geometric schemes*. In this case we will show that they define fractal sets having the Hausdorff dimension equal to the box-counting dimension. Moreover, the metric topology of these sets is completely described by an ultrametric, termed *net metric*, associated to the scheme in a natural way. The ultrametric contains the information about the scaling rates of the fractal set, with respect to the original metric. Moreover, this metric can replace the original metric for evaluating the distance between points close to each other. In concrete examples, where a resolution of the measures involved is always present, we can substitute the original metric by the net metric which, from a computational viewpoint is easier to use.

Falconer in [6], studied a simple renormalization condition defined in a dynamical way, which implies the coincidence of the Hausdorff dimension and the box-counting dimension. The uniformity conditon verified by a uniform geometric scheme can be also expressed, by saying that the distance between two points evalueated with respect to the original and the net metrics, have the same infinitesimal order. Thus the uniformity condition can be viewed as the simplest renormalization condition.

Uniform geometric schemes lead then to a construction of uniform Cantor-like sets which are ultrametric spaces. These kind of metrics satisfy some properties which are far from the common intuition, for instance, the radius of a ball coincides with its diameter. This perhaps can explain why fractals appear as "strange" sets.

Even if the uniformity condition that we use may seem rather strong, nevertheless, there are many examples of geometric construcitons which can be reduced to a uniform geometric scheme. Several exmples can be found in Falconer's recent book [7]. From this point of view we can see that the approach presented here unifies different ways of estimating the dimension of a large class of fractals. This framework, also includes the Iterated Function Systems of Barnsley and co-workers. In particular their results about the computation of the dimension in Barnsley and Demko [1] and Barnsley, Elton and Hardin [2], can be obtained as straightforward corollaries of the theorem 3.1 below.

2. Definitions. Our setting is a metric space (Ω, d) which we assume to be complete. In what follows we refer to the metric of this space as the basic metric.

Usually a recursive definition of fractal set is carried out with the help of a symbolic space whose elements are associated to the sets involved in the construction, in an arbitrary but specified manner. In most of the cases, from the topological properties of the symbolic space, we can deduce the topological properties of the limiting set obtained from the geometrical construction.

In this paper we only deal with two types of the most used symbolic spaces. They are defined as follows.

Let N be a natural number and denote by Σ^N the set of infinite sequences $\mathbf{i} = (i_1, i_2, \dots i_n \dots)$ with $i_j \in \{1, 2, \dots, N\}$. We say that $\mathbf{i}$ is a *multiindex*. For any multiindex $\mathbf{i}$ we denote by $\mathbf{i}|n$ the first finite sequence of length n included in the sequence $\mathbf{i}$. That is we set $i_1, i_2, \dots, i_n = \mathbf{i}|n$.

The definition of most of the classical fractals like the Cantor set, the von Koch's curve, the Keiswetter's curve and so on, are made up by means of the symbolic space Σ^N, where N^n is the number of sets involved in the n-th step of the recursive definition. More irregulars limit sets can be obtained if we replace the symbolic space Σ^N with its subspace Σ_M^N defined as follows.

Let M be a $N \times N$ matrix whose entries are 0 or 1.

We refer to M as a transition matrix. By means of the matrix M we can select some elements of Σ^N in the following way.

We say that a multiindex $\mathbf{i}$ is *M-admissible* if $M_{i_j, i_{j+1}} = 1, \forall j$.

The set Σ_M^N is defined as the subset of Σ^N formed by all the M-admissible multiindex. See [2], for an account of the several situations in which the symbolic space Σ_M^N may be used to define fractal sets.

In the sequel we will use Σ to denote the sets Σ^N and Σ_M^N, when the definitions or the properties involved concern both the symbolic spaces.

A *geometric scheme* is a family of compact subsets $(A_{\mathbf{i}|n})_{\mathbf{i}}$ of the metric space Ω indexed in the symbolic space Σ, which satisfies the following properties

$$\begin{array}{lll} \text{(i)} & A_{\mathbf{i}|n,j} \cap A_{\mathbf{i}|n,k} = \emptyset & \forall j \neq k \ \ \forall \mathbf{i} \forall n . \\ \text{(ii)} & A_{\mathbf{i}|n,j} \subseteq A_{\mathbf{i}|n} & \forall j \ \forall \mathbf{i} \ \forall n . \\ \text{(iii)} & \lim_{n \to +\infty} d(A_{\mathbf{i}|n}) = 0 & \forall \mathbf{i} \in \Sigma \end{array} \qquad (1)$$

We call *kernel* of the geometric scheme $(A_{\mathbf{i}|n})_{\mathbf{i}}$ the limit set $\mathcal{N}$ made up by means of the sets $A_{\mathbf{i}|n}$ in the following way:

$$\mathcal{N} = \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{i}|n} A_{\mathbf{i}|n} .$$

Generally $\mathcal{N}$ is a fractal and we say that $\mathcal{N}$ is a fractal of Cantor-type if the symbolic space Σ is Σ^N, while if the symbolic space is Σ_M^N we call $\mathcal{N}$ *fractal of Markov-type* or *recurrent fractal*.

We suppose that the transition matrix M is irreducible, that is there exists a $k > 0$ such that $(M^k)_{i,j} > 0 \ \forall i, j = 1, \ldots, N$. From this last hypothesis and the properties (1), it is easy to check that $\mathcal{N}$ is a compact, totally disconnected, perfect set. That is $\mathcal{N}$ is a Cantor set. We note that the irreducibility hypothesis for the matrix M is necessary in order to guarantee these topological properties.

For the dimension estimates we need some information about the metric properties of the set $\mathcal{N}$. To this end we introduce them in the following way.

Let $(h_1, \ldots, h_N)$ and $(k_1, \ldots, k_N)$ be positive real numbers less than 1. We suppose that the following *metric* relations hold.

$$0 < h_j \cdot |A_{\mathbf{i}|n}| \leq |A_{\mathbf{i}|n,j}| \leq k_j \cdot |A_{\mathbf{i}|n}| \,, \tag{2}$$

where $|\cdot|$ denote the diameter expressed with respect to the basic metric of the space Ω. We say that a geometric scheme $(A_{\mathbf{i}|n})_{\mathbf{i}\in\Sigma}$ satisfying (2), verifies the *inequality scaling property* whenever there is a j such that $h_j < k_j$ and that it verifies the *equality scaling property* if $h_j = k_j$ for all $j = 1, \ldots, N$.

Further, we suppose that the geometric scheme $(A_{\mathbf{i}|n})_{\mathbf{i}}$ satisfies a uniformity metric condition, namely, the following

$$0 < s \cdot |A_{\mathbf{i}|n}| \leq d(A_{\mathbf{i}|n,j}, A_{\mathbf{i}|n,k}) \quad j \neq k \tag{3}$$

for some $s > 0$ and for all $\mathbf{i} \in \Sigma$.

We call *uniform geometric schemes* the schemes that satisfy the above conditions (2) and (3).

Now we are ready to point out some topological properties of such geometric constructions. We will introduce a metric that is rather natural in our context. For any curtailed multiindex $\mathbf{i}|n = i_1, \ldots, i_n$, we put $d_{i_1,\ldots,i_n} = |A_{i_1,\ldots,i_n}|$. The family $(d_{i_1,\ldots,i_n})_{\mathbf{i}\in\Sigma}$ is decreasing and converging to zero.

For any x and $y \neq x$ belonging to $\mathcal{N}$ we can find a unique multiindex $\mathbf{i} \in \Sigma$ with the following properties.

(i) $x, y \in A_{\mathbf{i}|n}$; (ii) $\exists j$ such that $x \in A_{\mathbf{i}|n,j}$ and $\exists k \neq j$ such that $y \in A_{\mathbf{i}|n,k}$. Observe that $A_{\mathbf{i}|n,j} \cap A_{\mathbf{i}|n,k} = \emptyset$.

We set $D(x,y) = d_{i_1,\ldots,i_n}$ and, for $x = y$ we define $D(x,y) = 0$; in this way we get a new metric on the set $\mathcal{N}$. Indeed if $x \neq y$ for all $z \in \mathcal{N}$

there exists two indexes $(i_1, \dots, i_n)$ and $(j_1, \dots, j_m)$ with the properties stated above. That is: $x, z \in A_{\mathbf{i}|n}$ and $z, y \in A_{\mathbf{j}|m}$. Since $z \in A_{\mathbf{i}|n} \cap A_{\mathbf{j}|m}$ by property (1) it will be $A_{\mathbf{j}|m} \subseteq A_{\mathbf{i}|n}$ or $A_{\mathbf{i}|n} \subseteq A_{\mathbf{j}|m}$; suppose that the latter holds. Then we have $j_1, \dots, j_m = i_1, \dots, i_m$ and $x, y \in A_{i_1,\dots,i_n}$. So we get:

$$d_{i_1,\dots,i_n} = D(x,y)' \le \max(d_{i_1,\dots,i_n}, d_{i_1,\dots,i_m}) = \max(D(x,z), D(z,y)) \ .$$

Thus the distance D is indeed a metric. In particular as the last relation shows D is a ultrametric.

Let us call this metric D, *net metric*. This term is justified because the family of sets belonging to a geometric scheme is a net.

A net metric is a suitable choice in our setting, in fact it can be shown that the topology of the kernel of a geometric scheme is compatible with the net metrics, having as ranges, schemes of positive real numbers $\{b_{\mathbf{i}|n}\}_{\mathbf{i}}$ in which for all $\mathbf{i}$ the sequences $(b_{\mathbf{i}|n})_n$ are decreasing. In particular, if the geometric scheme is uniform, then the basic metric on the kernel $\mathcal{N}$ is equivalent to the net metric associated with the scheme.

Let us check now this last claim. Indeed, observe that for all $x, y \in \mathcal{N}$

$$d(x,y) \le d_{i_1,\dots,i_n} = D(x,y)$$

holds, where $(i_1, \dots, i_n)$ is the multiindex such that:

$$x, y \in A_{i_1,\dots,i_n} \ , \quad x \in A_{i_1,\dots,i_n,j} , \quad y \in A_{i_1,\dots,i_n,k} \quad with\ j \ne k \ .$$

If we put $S(x,y) = d(A_{i_1,\dots,i_n,j} A_{i,\dots,i_n,k})$ and call $S(x,y)$ the *separation* of the two points x and y, we have:

$$0 < s \cdot D(x,y) \le S(x,y) \le d(x,y) \ .$$

And thus for $x \ne y : s \cdot D(x,y) \le d(x,y) \le D(x,y)$ hold. Let us summarize these properties.

PROPOSITION 2.1. *Let $(\mathcal{N}, d)$ be the kernel of a uniform geometric scheme. Then $(\mathcal{N}, D)$ is an ultrametric space metrically equivalent to $(\mathcal{N}, d)$.*

The net metric D records the scaling properties evaluated with respect to the basic metric d, which contains the essential information we need to estimate the dimension of the set $\mathcal{N}$. Further, for two points x, y close each other, $D(x, y)$ is nearly equal to $d(x, y)$. This means that when we are dealing with concrete applications, we can replace the basic metric of the space of our problem with the net metric D which possesses the most suitable form to be computationally treated.

The characteristic properties of an ultrametric space seem to be very strange if we compare them with the corresponding ones, for instance, of euclidean spaces. Among these we will recall the following:

(i) The diameter of a ball coincides with its radius.

(ii) If two balls intersect then one ball is contained in the other.

(iii) A ball is open if and only if it is closed.

Moreover the sets of the scheme form a countable basis for the kernel which then is a zero-dimensional compact polish-space.

Fractals, generally, have a intricate detailed structure at every scale and a way for "measure" their geometrical complexity is to estimates their Hausdorff dimension. Let us recall now, the definition.

For any set $A \subset \Omega$ and $\alpha > 0$ the α-auxiliary Hausdorff measures are defined as follows. For $\delta > 0$ define

$$\mathcal{H}^{\alpha}_{\delta}(A) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^{\alpha} \middle| A \subseteq \bigcup_{i=1}^{\infty} U_i, U_i \text{ open } |U_i| \leq \delta \right\} .$$

For any $\delta > 0$, $\mathcal{H}^{\alpha}_{\delta}$ is an outer measure. The α-dimensional Hausdorff measure of A is given by

$$\mathcal{H}^{\alpha}(A) = \sup_{\delta > 0} \mathcal{H}^{\alpha}_{\delta}(A) = \lim_{\delta \to 0^+} \mathcal{H}^{\alpha}_{\delta}(A) ,$$

obtaining this way a regular metric outer measure. Standard references on Hausdorff measures and dimensions are the book of Falconer [5] and that of Rogers [12].

The positive real number α such that

$$\alpha = \inf \{\beta | \mathcal{H}^{\beta}(A) = 0\} ,$$

is said to be the Hausdorff dimension of the set A and, is denoted by $dim_H(A)$. It can be shown that

$$\alpha = \sup\{\beta | \mathcal{H}^{\beta}(A) = +\infty\} .$$

According to Falconer [5] we say that A is a α-set, if in the critical value α the α-dimensional Hausdorff measure of A is positive and finite.

Observe that if A is an α-set then $dim_H(A) = \alpha$, but the converse is not generally true.

In many cases the Hausdorff dimension is difficult to estimate. For this reason it has been introduced another kind of dimension, called *box-counting dimension.* It is largely used because of its easier computational estimate.

We recall that the *lower* and *upper* box-counting dimension of a set A are defined by

$$\underline{dim}_B(A) = \liminf_{\epsilon \to 0^+} \frac{\log N(\epsilon)}{-\log \epsilon}$$

and

$$\overline{dim}_B(A) = \limsup_{\epsilon \to 0^+} \frac{\log N(\epsilon)}{-\log \epsilon} ,$$

where $N(\epsilon)$ is the least number of sets of diameter at most ϵ which are needed to cover A. By the definitions easily follows

$$\underline{dim}_B(A) = \inf \{s | \liminf_{\epsilon \to 0} \epsilon^s N(\epsilon) < +\infty\} ,$$

$$\overline{dim}_B(A) = \inf \{s | \limsup_{\epsilon \to 0} \epsilon^s N(\epsilon) < +\infty\} .$$

Thinking of $\epsilon^\alpha N(\epsilon)$ as α-evaluations of a cover of A we, immediately, get

$$dim_H(A) \leq \underline{dim}_B(A) \leq \overline{dim}_B(A) .$$

If the lower and upper box-counting dimensions of A coincide, we say that their common value is the *box-counting dimension* of the set A and we write it as $dim_B(A)$.

Both the Hausdorff and box-counting dimensions are metrically invariant. Thus the proposition 2.1 enables us to replace the basic metric d with the net metric D in the dimension estimates. For this purpose we need the following well-known theorem about non-negative matrices.

Recall that a matrix is said to be non-negative if all its entries are non-negative numbers.

THEOREM 2.2. (*Perron-Frobenius*) (Seneta [13]). *Let M be any $N \times N$ non-negative irreducible matrix. Then the following properties hold.*

(i) There exist a positive eigenvalue $\lambda(M)$.
(ii) The left and right eigenvectors associated with $\lambda(M)$ *are strictly positive.*
(iii) $|\lambda| < \lambda(M)$ *for any other eigenvalue* λ *of* M.
(iv) The dimension of the eigenspace associated with $\lambda(M)$ *is* 1.
(v) $\min_{i=1,\dots,N} \sum_{j=1}^{N} M_{i,j} \leq \lambda(M) \leq \max_{i=1,\dots,N} \sum_{j=1}^{N} M_{i,j}$ *and* $\min_{j=1,\dots,N} \Sigma_{i=1}^{N} M_{ij} \leq \lambda(M) \leq \max_{j=1,\dots,N} \Sigma_{i=1}^{N} M_{i,j}$.

The positive number $\lambda(M)$ is also called the *Perron-Frobeniuus* eigenvalue of M.

We will consider now Markov-type schemes having M as irreducible transition matrix. The metric inequalities (2) enable us to define other two net metrics made up from the lower and upper scaling rates. With H^α and K^β we denote the following matrices:

$$H^\alpha = \begin{pmatrix} h_1^\alpha & 0 & \dots & 0 \\ 0 & h_2^\alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & h_N^\alpha \end{pmatrix} \quad K^\beta = \begin{pmatrix} k_1^\beta & 0 & \dots & 0 \\ 0 & k_2^\beta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k_N^\beta \end{pmatrix}. \qquad (4)$$

Let us choose the left eigenvectors $(v_1, \dots, v_N)$ and $(w_1, \dots, w_N)$ corresponding to the Perron-Frobenius eigenvalues of the matrices MH^α and MK^β; suppose further that the eigenvectors are normalized, that is $\sum_{i=1}^{N} v_i = 1$, and $\sum_{i=1}^{N} w_i = 1$.

Let us denote by D_L be the net metric whose range is the following scheme:

$$\{h_{i_1} \dots h_{i_n} \cdot v^{1/\alpha}\}_{\mathbf{i}_n \in \Sigma_M^N},$$

and by D_U the net metric whose range is:

$$\{k_{i_1} \dots k_{i_n} \cdot w_{i_n}^{1/\beta}\}_{\mathbf{i} \in \Sigma_M^N}.$$

We call D_L and D_U the *lower* and *upper net metrics* associated with the Markov-type scheme $(A_{\mathbf{i}|n})_{\mathbf{i} \in \Sigma_M^N}$. The metric conditions (2) imply:

$$c \cdot D_L(x,y) \leq D(x,y) \leq C \cdot D_U(x,y), \quad \forall x\, y \in \mathcal{N}, \qquad (5)$$

with c and C suitable constants. To simplify the calculations we define two new net metrics as follows. We put $D'_L := c \cdot D_L$ and $D'_U := C \cdot D_U$.

Our next step is to evaluate the Hausdorff dimension of $\mathcal{N}$ with respect to D'_L and D'_U.

3. Dimensions estimates. Now, we are in position to state and prove the property about the Hausdorff dimension of $\mathcal{N}$. The following theorem generalizes to recurrent uniform Cantor sets the dimension estimates, which are well-known in the case of self-similar sets. The proof is carried out by elementary technics, using in the crucial part of the estimates, the upper and lower net metrics. We observe that the geometric approach that we use, simplifies considerably the method of proof.

THEOREM 3.1. *Let $\mathcal{N}$ be the kernel of a geometric scheme of Markov-type $(A_{\mathrm{i}|n})_{\mathrm{i}\in\Sigma_M^N}$ with M as irreducible transition matrix. Then we have*

$$\alpha \leq dim_H(\mathcal{N}) = dim_B(\mathcal{N}) \leq \beta \, ,$$

provided that α and β satisfy the following equations:

$$\lambda(MH^{\alpha}) = 1 \quad \lambda(MK^{\beta}) = 1 \, ,$$

where H^{α} and K^{β} are the diagonal matrices in (4).

Proof. Since the metric d is equivalent to the net metric D we may use this last metric to evaluate the Hausdorff dimension of $\mathcal{N}$.

To obtain the estimate from above we may evaluate the β-dimensional Hausdorff measure of $\mathcal{N}$ with respect to the net metric D'_U.

For any covering of $\mathcal{N}$ made by the sets of the given scheme, we have:

$$\sum_{j:(i_1,\ldots,i_n,j)\in M-adm} D'_U(A_{i_1,\ldots,i_n,j})^{\beta} = D'_U(A_{i_1,\ldots,i_n})^{\beta} \cdot \sum_{j:(i_1,\ldots,i_n,j)\in M-adm} k_j^{\beta} \cdot \frac{w_j}{w_{i_n}}$$

$$= D'_U(A_{i_1,\ldots,i_n})^{\beta} \cdot \sum_{j=1}^{N} M_{i_n,j} k_j^{\beta} \frac{w_j}{w_{i_n}} = D'_U(A_{i_1,\ldots,i_n})^{\beta} \cdot \lambda(MK^{\beta}) \frac{w_{i_n}}{w_{i_n}} \, .$$

Thus:

$$\sum_{j:(i_1,\ldots,i_n,j)\in M-adm} D'_U(A_{i_1,\ldots,i_n,j})^{\beta} = D'_U(A_{i_1,\ldots,i_n})^{\beta} \, , \qquad (6)$$

for all $\mathbf{i} \in \Sigma_M^N$ and for all n.

For any $\epsilon > 0$ we can always find a n' such that $D'_U(A_{i_1,\dots,i_n}) \leq \epsilon\, \forall n \geq n'$ and, using inductively the equality (6), we get a finite β-evaluation of $\mathcal{N}$. Then since $D(x,y) \leq D'_U(x,y)\ \forall x,y \in \mathcal{N}$ we have $dim_H(\mathcal{N}) \leq \beta$.

For the estimate from below we first have to check that for any open cover, say $\{U\}$ of $\mathcal{N}$ we can find a cover of $\mathcal{N}$ made by sets of the scheme $(A_{\mathbf{i}|n})_{\mathbf{i}\in\Sigma_M^N}$ whose associated α-evaluation is not greater than the α-evaluation associated with the cover $\{U\}$.

In fact since the range of the net metric D without the point 0 is a discrete subset of $\mathbf{R}$ the diameter of any $U \in \{U\}$ is always attained. Then for any given $U \in \{U\}$ we can find a $A_{\mathbf{i}|n}$ such that $U \subseteq A_{\mathbf{i}|n}$ and $D(U) = D(A_{\mathbf{j}|m})$. Then:

$$\sum_{U\in\{U\}} D(U)^\alpha = \sum_{\mathbf{j}|m} D(A_{\mathbf{j}|m})^\alpha .$$

Moreover, we have

$$\sum_{\mathbf{i}|n} D'_L(A_{\mathbf{i}|n})^\alpha \leq \sum_{\mathbf{j}|m} D(A_{\mathbf{j}|m})^\alpha \leq \sum_{U\in\{U\}} D(U)^\alpha ,$$

where in the first inequality $(A_{\mathbf{i}|n})_{\mathbf{i}|n}$ is a disjoint subfamily of $(A_{\mathbf{j}|m})_{\mathbf{j}|m}$ that covers the kernel $\mathcal{N}$.

In the same way as above it can be shown that:

$$\sum_{j:(i_1,\dots,i_n,j)\in M-adm} D'_L(A_{i_1,\dots,i_n,j})^\alpha = D'_L(A_{i_1,\dots,i_n})^\alpha , \qquad (7)$$

holds for all $\mathbf{i} \in \Sigma_M^N$ for all n; an inductive use of this last equality together with the following relation $D'_L(x,y) \leq D(x,y)\ \forall x,y \in \mathcal{N}$, enable us to conclude with:

$$\alpha \leq dim_H(\mathcal{N}) .$$

Since $dim_H(\mathcal{N}) \leq \overline{dim}_B(\mathcal{N})$, to reach the claim, we have only to show that $\overline{dim}_B(\mathcal{N}) \leq dim_H(\mathcal{N})$.

Let us choose $\epsilon > 0$. For any index $i \in \Sigma_M^N$ there exists a $n(i) \in \mathbf{N}$ such that

$$h\epsilon < d_{i_1,\dots,i_n(i)} \leq \epsilon \quad \text{with} \quad h = \min\{h_1,\dots,h_N\} . \qquad (8)$$

We can choose elements of the scheme $(A_{\mathrm{i}|n})_{\mathrm{i}\in\Sigma_M^N}$ in such a way that they form a cover of $\mathcal{N}$ and their diameters satisfy the relations (8). For any ϵ we denote by $M(\epsilon)$ the number of such sets. We get

$$h^{\gamma}\cdot N(\epsilon)\epsilon^{\gamma}\leq h^{\gamma}\cdot M(\epsilon)\epsilon^{\gamma}\leq\sum(d_{i_1,\dots,i_{n(\mathrm{i})}})^{\gamma}=$$

$$=\sum D(A_{i_1,\dots,i_{n}(\mathrm{i})})^{\gamma},$$

where $\gamma = dim_H(\mathcal{N})$. One checks, immediately, that the indexes satisfying (8) corresponds to elements of $(A_{\mathrm{i}|n})_{\mathrm{i}\in\Sigma_M^N}$ that do not intersect. By the preceding step of the proof we get:

$$h^{\gamma}\cdot N(\epsilon)\epsilon^{\gamma}\leq\sum_{i_1,\dots,i_{n(\mathrm{i})}}(d_{i_1,\dots,i_{n}(\mathrm{i})})^{\beta}\leq\sum_{i_1,\dots,i_{n}(\mathrm{i})}D'_U(A_{\mathrm{i}|n(\mathrm{i})})^{\beta}<+\infty,$$

hence:

$$\underline{dim}_B(\mathcal{N})\leq\overline{dim}_B(\mathcal{N})\leq\gamma=dim_H(\mathcal{N}).$$

The proof is complete.

If the geometric scheme $(A_{\mathrm{i}|n})_{\mathrm{i}\in\Sigma_M^N}$ is of Cantor-type, that is $\Sigma=\Sigma^N$, we have that the matrix M has all its entries equal to 1. In this case the matrix MH^{α} becomes:

$$MH^{\alpha}=\begin{pmatrix} h_1^{\alpha} & h_2^{\alpha} & \dots & h_N^{\alpha}\\ h_1^{\alpha} & h_2^{\alpha} & \dots & h_N^{\alpha}\\ \vdots & \vdots & \ddots & \vdots\\ h_1^{\alpha} & h_2^{\alpha} & \dots & h_N^{\alpha}\end{pmatrix}.$$

And by part (v) of Theorem 2.2 the equation $\lambda(MH^{\alpha}) = 1$ becomes the following well-known Moran's formula $\sum_{j=1}^{N} h_j^{\alpha} = 1$.

Thus we have proved the following result which, as in [3], shows the validity of Moran's formulas, also in non-dynamical settings.

COROLLARY 3.2. *Let $\mathcal{N}$ be the kernel of a Cantor-type scheme satisfying the scaling metric relations (2). Then we have:*

$$\alpha\leq dim_H(\mathcal{N})=dim_B(\mathcal{N})\leq\beta,$$

provided that α and β are solutions of the following equations:

$$\sum_{j=1}^{N} h_j^{\alpha} = 1 \quad \sum_{j=1}^{N} k_j^{\alpha} = 1 \ .$$

Now, let us consider the case in which for each i and j we have, $h_i = h_j$. In this case the equation $\lambda(MH^{\alpha})$ becomes:

$$1 = \lambda(M \cdot h^{\alpha} \cdot Id) = \lambda(h^{\alpha} \cdot M) = h^{\alpha}\lambda(M)$$

and then, we obtain

$$\alpha = \frac{\log \lambda(M)}{|\log h|} \ .$$

So we get the following:

COROLLARY 3.3. *Let $\mathcal{N}$ be the kernel of a geometric scheme of Markov-type verifying the scaling relations (2). Suppose further that for each i and j $h_i = h_j$ and $k_i = k_j$ hold.*

Then we have:

$$\frac{\log \lambda(M)}{|\log h|} \leq dim_H(\mathcal{N}) = dim_B(\mathcal{N}) \leq \frac{\log \lambda(M)}{|\log k|} \ .$$

Although the uniformity condition (3) may seem rather strong it is satisfied in many concrete examples as the recent book by Falconer [7] shows.

We have seen that in the case of schemes satisfying the uniformity condition (3) what is really involved in the estimate of the dimensions of their kernels are the metric features and not the covering properties.

A widely studied case, which can be reduced to this situation, concerns the Iterated Function Systems (ISF, for short) of Barnsley and co-workers. See [1].

4. Geometric schemes dynamically defined. A *Iterated Function System* is a finite set of maps T_j with $T_j : \Omega \to \Omega \quad j = 1, \ldots, N$, which, together with their inverses are supposed to be contractions verifying the inequalities:

$$h_j d(x,y) \leq d(T_j x, T_j y) \leq k_j d(x,y) \quad j = 1, \ldots, N \ .$$

Following the classical approach of Hutchinson [8], we can define a map $T : \mathcal{K}(\Omega) \to \mathcal{K}(\Omega)$ in this way:

$$T(B) = \bigcup_{j=1}^{N} T_j(B)$$

where $\mathcal{K}(\Omega)$ is the set of all compact non-empty subsets of (Ω, d) endowed with the Hausdorff metric d_H. The Blaschke's Selection theorem (see Falconer [5]) ensures that if (Ω, d) is complete, then the metric space $(\mathcal{K}(\Omega), d_H)$ is complete too.

Given a IFS $\{T_1, \dots, T_N\}$ we can define a geometric scheme by means of the following well-known theorem.

THEOREM 4.1. (*Hutchinson* [8]). *Let* $\{T_1, \dots, T_n\}$ *be a IFS with ratios* $k_i < 1 \quad i = 1, \dots, N$. *Then there exists a non-empty compact set* $A \subseteq \Omega$ *such that*:

$$A = T(A) = \bigcup_{j=1}^{N} T_j(A) .$$

The geometric scheme associated with the above IFS is made up in the following way:

$$A_i = T_i(A) ,$$

$$A_{i_1,\dots,i_n} = T_{i_1} \circ T_{i_2} \circ \dots \circ T_{i_n}(A) . \qquad (9)$$

A IFS is called *disjoint* if $T_i(A) \cap T_j(A) = \emptyset$ for $i \neq j$.

We now consider the case in which the IFS is disjoint and whose maps are similitudes. Proceding as in Hutchinson [8], it is easy to see that $(A_{\mathbf{i}|n})_{\mathbf{i}}$ satisfies properties (1), with $A = \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{i}|n} A_{\mathbf{i}|n}$. Thus it is a geometric scheme. Moreover, one can easily check that:

$$|A_{i_1,\dots,i_n j}| = k_j \cdot |A_{i_1,\dots,i_n}| \quad \forall j = 1, \dots, N .$$

Then the scheme $(A_{\mathbf{i}|n})_{\mathbf{i}}$ satisfies the metric relations (2). In particular the range of the net metric D is $\{k_{i_1} \dots k_{i_n}\}_{i \in \Sigma}$.

If $x, y \in A$, the separation $S(x, y)$ is:

$$S(x, y) = d(A_{\mathbf{i}|nj}, A_{\mathbf{i}|nk}) ,$$

where $x \in A_{\mathbf{i}|nj}$, $y \in A_{\mathbf{i}|nk}$ and $A_{\mathbf{i}|nj} \cap A_{\mathbf{i}|nk} = \emptyset$. Thus:

$$S(x,y) = d(A_{\mathbf{i}|nj}, A_{\mathbf{i}|nk}) = d(T_{i_1} \dots T_{i_n} T_j(A),\ T_{i_1} \dots T_{i_n} T_k(A)) =$$

$$= k_{i_1} \dots k_{i_n} d(T_j(A),\ T_k(A)) .$$

Recalling that in the disjoint case $d(T_j(A), T_k(A)) > 0$ holds, we immediately, obtain:

$$0 < \inf \frac{k_{i_1} \dots k_{i_n} d(T_j(A), T_k(A))}{k_{i_1} \dots k_{i_n}} = \inf \frac{S(x,y)}{D(x,y)} .$$

This last relation shows that the geometric scheme induced by the disjoint IFS satisfies condition (3); then it is uniform.

Now for the estimates of disjoint IFS's we can apply Theorem 3.1 and its corollaries obtaining in this way well-known results by Barnsely, Elton and Hardin [2] and Bedford [4].

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