ONE DIMENSIONAL COLLOCATION AT GAUSSIAN POINTS AND SUPERCONVERGENCE AT INTERIOR NODAL POINTS (*)

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SOMMARIO. - *In questo lavoro vengono estesi e completati i risultati ottenuti da M. Bakker* [1] *sulla proprietà di superconvergenza in punti interni del metodo di Collocazione ai punti di Gauss. In particolare, sotto opportune ipotesi di regolarità (cfr. De Boor e Swartz* [2]*), vengono individuati in ogni intervallo della partizione tutti i punti in cui l'ordine di convergenza del metodo è superiore all'ordine ottimo di convergenza globale.*

SUMMARY. - *Here we extend and complete the results that M. Bakker* [1] *recently proved about a special kind of superconvergence of the method of Collocation at Gaussian points: the superconvergence at interior nodal points. We will prove that under the smoothness assumptions made by De Boor and Swartz in* [2] *there exist particular points inside each segment of the partition in which the rates of convergence are one order better than the optimal global ones.*

1980 Mathematics Subject Classification (1985 Revision). Primary 66N35, 33A65.

Introduction. Superconvergence at the knots is a well-known phenomenon [2] in the case of collocation method for approximately solving m-th order boundary value problems; it is known that using piecewise polynomial functions of degree k and class C^{m-1}, the approximation error and its first $m-1$ derivatives are $O(h^{2r})$, where h is the maximum subinterval length, $r = k + 1 - m$ is the number of collocation points on each

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(*) Pervenuto in Redazione il 22 ottobre 1990.

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subinterval, and $k \geq 2m-1$ is the degree of the finite element space.

Here we will study another kind of superconvergence, the superconvergence at interior nodal points, that has been recently pointed out by M. Bakker [1]: he proved that inside each subinterval there exist specific points where the rates of convergence of the solution and its derivative are one order better than the optimal ones, i.e. the approximation error e is $O(h^{k+2})$ and De is $O(h^{k+1})$.

In the next sections extending the work of Bakker, we will find in general that (§3) $D^{m-i}e \in O(h^{k+2-m+i})$ at the zeros of the Jacobi polynomial $P_{r-1}^{(i,i)}(t)$, shifted to each segment of the partition, for any i, $0 \leq i \leq m$ and $k \geq 2m$. These results, already known in the case of two-point boundary value problems [3],[4], will be proved in this paper in a different way and for more general boundary conditions.

Moreover, we will also see (§4) that the Jacobi ones are the only interior points of superconvergence, that is it can't exist any other point of superconvergence but nodal and Jacobi ones, completing in this way the results in [1] ($i = m,\ m-1$).

2. Collocation at Gaussian points.

Consider the m-th order boundary problem

$$Lu = D^m u + \sum_{i=0}^{m-1} p_i(x) D^i(u) = f(x),\ x \in I = [a,b] \qquad (2.1a)$$

$$B_i[u] = 0, \quad i = 1,\dots,m, \qquad (2.1b)$$

where B_i, $i = 1,\dots,m$, are continuous linear functionals over $C^{m-1}(I)$ and assume that p_i, $i = 0,\dots,m$, and f are sufficiently smooth functions, that B_i, $i = 1,\dots,m$, are linearly independent over the space $P_{m-1}(I)$ of polynomials of degree $\leq m-1$, and assume that the problem (2.1) has a unique solution. Let

$$\Delta : a = x_0 < x_1 < \dots < x_N = b,$$
$$I_j = [x_{j-i}, x_j], \quad h_j = x_j - x_{j-1}, \quad j = 1,\dots,N,$$
$$h = \max_j h_j$$

be a partition of I. Then given an integer $k \geq 2m-1$ define the finite element space S_0 by:

$$S_0 = S_0(k, m-1, \Delta) = \{V \in C_0^{m-1} : V \in P_k(I_j),\ j = 1, \ldots, N\},$$

where

$$C_0^{m-1}(I) = \{V \in C^{m-1}(I) : B_i[v] = 0, \quad i = 1, \ldots, m\}.$$

Let

$$x_{js}^0 = x_{j-1} + (1 - t_s)h_j/2, \quad j = 1, \ldots, N,\ s = 1, \ldots, r,$$

denote the Gaussian points, where t_s, $s = 1, \ldots, r$, are the zeros of the r-th degree Legendre polynomial $P_r^{(0,0)}(t)$ with $t \in [-1, 1]$ and $r = k+1-m$. Under these assumptions De Boor-Schwarz [2] proved that there exists a unique solution $U \in S_0$ of the following collocation equations:

$$LU(x_{js}^0) = f(x_{js}^0),\ j = 1, \ldots, N\ s = 1, \ldots, r. \tag{2.2}$$

Furthermore, they showed that if $u \in C_0^{m+1}(I) \cap C^{2r+m}(I)$ solves the problem (2.1) then the error function $e = u - U$ has the following bounds:

$$||e||_i \leq C_1 h^{k+1-i},\ i = 0, \ldots, m, \tag{2.3}$$

$$|D^i e(x_j)| \leq C_2 h^{2r},\ i = 0, \ldots, m-1,\ j = 0, \ldots, N, \tag{2.4}$$

where:

$$||v||_0 = ||v||_{W^p(I)} = \sum_{i=0}^{p} ||D^i v||_{L_\infty(I)},$$

$$W^p(I) = \{v : D^i v \in L_\infty(I), \quad i = 0, \ldots, p\}$$

and C_1, C_2 are constants independent of h.

3. Superconvergence at interior nodal points. Let t_s^i, $s = 1, \ldots, r-i$, be the zeros of the Jacobi polynomial:

$$P_{r-i}^{(i,i)} = C_{r,i} \frac{D^{r-i}[(1-t^2)^r]}{(1-t^2)^i}, \quad i = 1, \ldots, m, \tag{3.1}$$

with $t \in [-1,1]$ and $C_{r,i}$ suitable constants depending only on r and i. These polynomials belong to a particular class of Jacobi ones and are known as Gegenbauer polnomials [5, page 102]; however in this work they will be always referred as Jacobi ones.

Let x^i_{js}, $s = 1,\dots,r-1$ be the Jacobi points defined by:

$$x^i_{js} = x_{j-1} + (1 + t^i_s)h_j/2 \ , j = 1,\dots,N \ .$$

THEOREM. *Let $u \in C_0^{m-1}(I) \cap C^{2r+m}(I)$ and $U \in S_0(k, m-1, \Delta)$ be the solutions of (2.1) and (2.2) respectively, where $k \geq 2m$ and Δ is a quasi-uniform partition of I, i.e. there exists $c > 0$ independent of h so that $h \leq ch_j$, for $j = 1,\dots,N$. If h is small enough then the error $e = u - U$ satisfies the following bounds:*

$$|D^{m-i}e(x^i_{js})| \leq C_i h^{k+2-m+i} \ ,$$

$$i = 0,\dots,m, \ j = 1,\dots,N, \ s = 1,\dots,r-i \ . \qquad (3.2)$$

This extends the results in [1] which correspond to the cases $i = m-1$ and $i = m$.

Proof. Since from (2.3) $D^{m-i} \in O(h^{k+1-m+i})$, it is sufficient to prove that there exist two functions f_i and g_i of order $O(1)$ in the space $C^{2r}(I_j)$ such that (for sake of simplicity we denote $P^{(i,i)}_{r-i}(t)$ with $P^i_{r-i}(t)$):

$$D^{m-i}e(x) = h^{k+2-m+i}g_i(x) + h^{k+1-m+i}(1-t^2)^i P^i_{r-i}(t) f_i(x) \ ,$$

$$i = 0,\dots,m, \ x \in I_j, \ t = 1 + 2(x - x_j)/h_j, \ j = 1,\dots,N \ . \qquad (3.3)$$

Induction will be used to get (3.3).

For $i = 0$, combining (2.1.a) and (2.3) leads to the following equalities:

$$\begin{aligned} D^m e(x) &= Le(x) - \sum_{i=0}^{m-1} p_i(x) D^i e(x) \\ &= h^{k+1-m}F(x) + h^{k+2-m}g_0(x) \ , \end{aligned}$$

with $F(x),\ g_0(x) \in \mathbf{O}(1)$ and from (2.2) $F(x) = P_r^0(t) f_0(x)$, where $t = 1+2(x-x_j)/h_j$ and $f_0(x) \in \mathbf{O}(1)$. Furthermore, the smoothness assumptions on u and $p_0, \dots, p_{m-1}$ yield $F_0(x), f_0(x), g_0(x) \in C^{2r}(I_j)$. This completes the proof in the case $i = 1$.

For $i > 0$ $(i \leq m)$, induction leads to the following equality:

$$D^{m-i+1}e(x) = h^{k+2-m+i-1} g_{i-1}(x) + \\ + h^{k+1-m+i-1}(1-t^2)^{i-1} P_{r-i+1}^{i-1}(t) f_{i-1}(x) \qquad (3.4)$$

where $f_{i-1}(x)$ and $g_{i-1}(x)$ are functions in $C^{2r}(I_j)$ of order $\mathbf{O}(1)$. Partial integration of (3.4) leads to

$$\begin{aligned} D^{m-i}e(x) = & D^{m-i}e(x_{j-1}) + h^{k+2-m+i-1} \int_{x_{j-1}}^{x} g_{i-1}(y)\,dy + \\ & - h^{k+1-m+i-1} \int_{x_{j-1}}^{x} f'_{i-1}(y) \int_{x_{j-1}}^{y} (1-t^2)^{i-1} P_{r-i+1}^{i-1}(t)\,dz\,dy + \\ & + h^{k+1-m+i-1} f_{i-1}(x) \int_{x_{j-1}}^{x} (1-t^2)^{i-1} P_{r-i+1}^{i-1}(t)\,dy\,, \qquad (3.5) \end{aligned}$$

while (3.1) and [6, formula 22.6.1] lead to

$$f_{i-1}(x) \int_{x_{j-1}}^{x} (1-t^2)^{i-1} P_{r+i-1}^{i-1}(t)\,dy = f_{i-1}(x) C_{r,i} (1-t^2)^i P_{r-i}^i(t) h_j/2;$$

hence $f_i(x) = C_{r,i} f_{i-1}(x) h_j/(2h)$, with $C_{r,i}$ depending only on r and i, $f_i(x) \in C^{2r}(I_j)$ and $f_i(x)$ of order $\mathbf{O}(1)$ since Δ is quasi-uniform.

Furthermore, using the bounds in (2.4), it can be proved that the other terms at the right hand side of (3.5) are $\mathbf{O}(h^{k+2-m+i})$; hence there exists a function $g_i(x)$ such that:

$$D^{m-i}e(x) = h^{k+2-m+i} g_i(x) + h^{k+1-m+i}(1-t^2)^i P_{r-i}^i(t) f_i(x)\,,$$

with $g_i(x) \in \mathbf{O}(1)$. Since $g_i(x) \in C^{2r}(I_j)$ (from (3.5) with the assumptions on f_{i-1} and g_{i-1}) this completes the proof.

4. Convergence rates outside Jacobi and nodal points. Finally, this work observes that the phenomenon of superconvergence cannot take place at any other point except Jacobi and nodal ones.

Suppose that there exists $x_{j0}^{m-i} \in (x_{j-1}, x_j)$ so that

$$\begin{aligned} & x_{j0}^{m-i} \neq x_{js}^{m-i}, \; s = 1, \ldots, r-m+i, \\ & D^i e(x_{j0}^{m-i}) \in \mathbf{O}(h^{k+2-i}), \end{aligned} \tag{4.1}$$

where i and j are two fixed integers $1 \leq i \leq m$ and $1 \leq j \leq N$; to simplify, replace x_{js}^{m-i} with y_s, $s = 0, \ldots, r-m+i$.

Given a function $f \in C^{m-i-1}(I_j)$ (if $i = m$ the space $C^{-1}(I_j)$ denotes $C^0(x_{j-1}, x_j)$) let $F(x) \in P_{k+1-i}(I_j)$ be its Hermite interpolant polynomial defined by:

$$\begin{aligned} & F(y_s) = f(x_s), && s = 0, \ldots, r-m+i, \\ & D^d F(x_{j-1}) = D^d f(x_{j-1}), && \\ & && d = 0, \ldots, m-i-1, \; 0 \leq i \leq m-1. \\ & D^d F(x_j) = D^d f(x_j), && \end{aligned} \tag{4.2}$$

It is known [7] that if $f \in C^{m+i-1}(I_j) \cap W^{k+2-i}(I_j)$, F can be represented as follows:

$$F(x) = \sum_{s=0}^{r-m+i} f(y_s) W_s(x) + \sum_{d=0}^{m-i-1} [D^d f(x_{j-1}) V_{d1}(x) + D^d f(x_j) V_{d2}(x)] \tag{4.3}$$

where $W_s(x)$, $V_{d1}(x)$ and $V_{d2}(x)$ are natural basis functions for Hermite interpolation and it holds:

$$||D^d(F - f)||_{L_\infty(I_j)} \leq C h^{k+2-i-d} ||D^{k+2-i} f||_{L_\infty(I_j)}, \tag{4.4}$$

for $0 \leq d \leq k+2-i$.

Since $D^i e \in C^{m-i-1}(I_j) \cap W^{k+2-i}(I_j)$, (4.3) and (4.4) lead to the following equality:

$$\begin{aligned} D^i e(x) = & \sum_{s=0}^{r-m+i} D^i e(y_s) W_s(x) \\ & + \sum_{d=0}^{m-i-1} [D^{d+i} e(x_{j-1}) V_{d1}(x) + D^{d+i} e(x_j) V_{d2}(x)] + \\ & + \mathbf{O}(h^{k+2-i}), \end{aligned}$$

which yields to

$$D^i e(x) \in \mathrm{O}(h^{k+2-i}),\ x \in I_j\ , \tag{4.5}$$

because: $D^i e(y_s) \in \mathrm{O}(h^{k+2-i})$ from (3.2) and (4.1);

$$D^{d+i}e(x_{j-1}),\ D^{d+i}e(x_{j-1}) \in \mathrm{O}(h^{2r}) \text{ from } (2.5);$$

$$W_s(x) \in \mathrm{O}(1),\ V_{d1}(x),\ V_{d2}(x) \in \mathrm{O}(h^d),\ x \in I_j\ .$$

From (4.5) it follows that

$$\inf_{V \in P_k(I_j)} ||D^i(u - V)||_{L_\infty(I_j)} \in \mathrm{O}(h^{k+2-i})\ , \tag{4.6}$$

where $u \in C^{2r+m}(I_j)$ solves a problem like (2.1); hence the estimate (4.6) holds for any function $u \in C^\infty(I_j)$. Consider for example the following function:

$$g(x) = \begin{cases} (h_j/2)^{k+1} f(1 + 2(x - x_j)/h_j), & x \in I_j, \\ 0 & x \in I \backslash I_j, \end{cases}$$

where $f(t) = e^{-t^2/(1-t^2)}$, with $t \in [-1, 1]$; then the results in [8, §6.6] may be used to prove the existence of two positive constants C and $C(f)$ independent of h such that:

$$\inf_{V \in P_k(I_j)} ||D^i(g - V)||_{L_\infty(I_j)} \geq C h_j^{k+1-i} ||D^{k+1} g||_{L_\infty(I_j)}$$
$$\geq C(f) h^{k+1-i}\ .$$

But this bound is in contradiction with (4.6) and (4.6) derives from (4.1), i.e. from the assumption of superconvergence at an interior nodal point in addition to Jacobi ones.

REMARK. The interior superconvergence property of collocation at Gaussian points can be proved (see [9, chapter 7]) for other classes of problems. For instance, using [2, Theorem 3.1] it is possible to extend the results here obtained in the linear case to m-th order non linear equations; while proceeding as in section 3 it can be proved that the estimates (3.2) are still true when solving a first order system of boundary value problems (see [10] for a proof of optimal rates of convergence and superconvergence at the knots in this case).

Acknowledgements. The author wishes to thank Professor A. Pasquali for many helful discussions.

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