VARIATIONS ON A THEME BY SCHEFFÉ (*)

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SOMMARIO. - *Una versione finitamente additiva del classico teorema di Scheffé suggerisce una versione rafforzata dello stesso teorema, nella quale la convergenza q.o. delle densità è sostituita dalla convergenza in misura.*

SUMMARY. - *We present here a finitely additive version of Scheffé's theorem. This version, in its turn, suggests a strengthened form of the traditional Scheffé's theorem, in which convergence a.e. of the densities is replaced by convergence in measure.*

We present here a finitely additive version of Scheffé's theorem [4]. This version is, in turn, used in order to strengthen Scheffé's theorem, by replacing the assumption of convergence a.e. of the densities by that of their convergence in measure.

Let $(\Omega, \mathcal{F}, \upsilon)$ denote a positive charge space, viz. υ is a positive charge (= finitely additive measure) defined on the field $\mathcal{F}$ of subsets of Ω. If $\int f d\upsilon$ denotes the usual D-integral $D\int f d\upsilon$, let $D(\upsilon) := \{f \in L^1(\upsilon) : f \geq 0, \int f d\upsilon = 1\}$ be the set of probability densities with respect to υ (see [1] for the definition of D-integral and of $L^1(\upsilon) := L_1(\Omega, \mathcal{F}, \upsilon)$ for charges. We shall adopt the same notation of that book).

By $f \cdot \upsilon$ with $f \in D(\upsilon)$ we shall denote the charge defined for every $A \in \mathcal{F}$ by

$$(f \cdot \upsilon)(A) := \int_A f d\upsilon .$$

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Then we have the following finitely additive version of Scheffé's theorem

(1) THEOREM. *For a sequence* $\{f_n\}$ *of elements of* $D(\upsilon)$ *and* $f \in D(\upsilon)$, *the following statements are equivalent:*

(a) $\{f_n\}$ *converges to* f *hazily;*

(b) $\lim_n \|f_n - f\|_1 = \lim_n \int |f_n - f| d\upsilon = 0$;

(c) $\lim_n \sup_{A\in\mathcal{F}} |\mu_n(A) - \mu(A)| = 0$, *where* $\mu_n := f_n \cdot \upsilon$ *and* $\mu := f \cdot \upsilon$.

Proof. *(a)* ⇒ *(b)* Set $\phi_n := f - f_n$ so that $\phi_n \to 0$ hazily; moreover $\int \phi_n d\upsilon = 0$, whence $\int \phi_n^+ d\upsilon = \int \phi_n^- d\upsilon$. Therefore $\int |f_n - f| d\upsilon = 2\int \phi_n^+ d\upsilon$. But $0 \leq \phi_n^+ \leq f$ and ϕ_n^+ is (T_1)-measurable for every $n \in \mathbf{N}$ and so the finitely additive version of Lebesgue dominated convergence theorem ([1], 4.6.14) yields $\lim_n \int |f_n - f| d\upsilon = 0$.

(b) ⇒ *(a)* This follows from Theorem 4.6.10 *(i)* of [1]. The proof of this theorem is based on Čebyšev's inequality which holds also in a finitely additive context.

(b) ⇒ *(c)* This follows easily on noting that for every $A \in \mathcal{F}$ one has

$$|\mu_n(A) - \mu(A)| = |\int_A f_n d\upsilon - \int_A f d\upsilon| \leq \int_A |f_n - f| d\upsilon \leq$$

$$\leq \int |f_n - f| d\upsilon$$

(see [1], 4.4.13 *(ii)*, *(iii)* and *(xi)*).

(c) ⇒ *(b)* Define a sequence of real charges on $\mathcal{F}$ via

$$\lambda_n(A) := \int_A (f_n - f) d\upsilon \; (A \in \mathcal{F},\ n \in \mathbf{N}) .$$

In view of theorem 4.4.13 *(xi)* in [1], conditions *(b)* and *(c)* read respectively

(b') $\lim_n |\lambda_n|(\Omega) = 0$,

and

(c') $\lim_n \sup_{A\in\mathcal{F}} |\lambda_n(A)| = 0$.

Now, by theorem 2.2.4 in [1], one has, for every $n \in \mathbf{N}$

$$0 \leq |\lambda_n|(\Omega) \leq 2 \sup_{A \in \mathcal{F}} |\lambda_n(A)| ,$$

hence *(c') ⇒ (b')* or, equivalently, *(c) ⇒ (b)*. ■

(2) COROLLARY. *Let Ω be a topological space, $\mathcal{F}$ a field containing the open sets and υ a positive charge on $\mathcal{F}$. If the sequence $\{f_n\}$, with $f_n \in D(\upsilon)$ for every $n \in \mathbf{N}$, converges hazily to $f \in D(\upsilon)$, then the sequence of probability charges $\{\mu_n := f_n \cdot \upsilon\}$ converges weakly to the probability charge $\mu := f \cdot \upsilon$, i.e.*

$$\lim_n \int \phi d\mu_n = \int \phi d\mu$$

for every continuous bounded $\phi : \Omega \to \mathbf{R} (\phi \in C_b(\Omega))$.

Proof. Let $\phi : \Omega \to \mathbf{R}$ be a continuous bounded function and let M be such that $|\phi| < M$. For every $x \in \mathbf{R}$, the set $S_x := \varphi^{-1}(\{x\})$ is closed and hence S_x is in $\mathcal{F}$.

There is at most a countable number of values of x for which $\upsilon(S_x) > 0$. Therefore for every $\varepsilon > 0$ one can choose $x_0, x_1, \ldots, x_r$ in $\mathbf{R}$ so that the following conditions are fulfilled

$$-M = x_0 < x_1 < \ldots < x_r = M ,$$

$$x_i - x_{i-1} < \varepsilon \quad (i = 1, 2, \ldots, r) ,$$

$$\upsilon(S_{x_i}) = 0 \quad (i = 0, 1, \ldots, r) .$$

Let $A_i := \varphi^{-1}(]-\infty, x_i[)$, $(i = 0, 1, \ldots, r)$; since A_i is open, it is in $\mathcal{F}$ for every i. Thus, by Theorem (1)(c), $\mu_n(A_i) \to \mu(A_i)$ and $\mu_n(A_i - A_{i-1}) \to \mu(A_i - A_{i-1})$ for every i. If I_i is the indicator of $A_i - A_{i-1}$, define $\phi_\varepsilon : \Omega \to \mathbf{R}$ by

$$\phi_\varepsilon := \sum_{i=1}^{r} x_i I_i .$$

Then $|\phi - \phi_\varepsilon| < \varepsilon$ and

$$\int \phi_\varepsilon d\mu_n = \sum_{i=1}^{r} x_i \mu_n(A_i - A_{i-1}) \to \sum_{i=1}^{r} x_i \mu(A_i - A_{i-1}) = \int \phi_\varepsilon d\mu .$$

Now

$$\left|\int \phi d\mu_n - \int \phi d\mu\right| \leq \left|\int (\phi - \phi_\varepsilon) d(\mu_n - \mu) + \left|\int \phi_\varepsilon d(\mu_n - \mu)\right| \leq \right.$$

$$\leq 2\varepsilon + \left|\int \phi_\varepsilon d\mu_n - \int \phi_\varepsilon d\mu\right|$$

which proves the assertion. ■

The previous proof is adapted from [3] (theorem 4.2)

A result similar to Corollary (2) (but in a different context) holds when υ is a measure on a suitable σ-algebra $\mathcal{B}$.

(3) THEOREM. *Let Ω be a separable metric space, $\mathcal{B}$ the Borel σ-algebra of Ω and υ a measure on $\mathcal{B}$. If the sequence $\{f_n\}$ of probability densities with respect to υ, i.e. $f_n \in D(\upsilon)$, converges in υ-measure to $f \in D(\upsilon)$, then the sequence of probability measures $\{\mu_n := f_n \cdot \upsilon\}$ converges weakly to the probability measure $\mu := f \cdot \upsilon$.*

Proof. Let $\{\mu_{n(k)}\}$ be any subsequence of $\{\mu_n\}$ and consider the corresponding subsequence of densities $\{f_{n(k)}\}$; this latter converges in υ-measure to f and therefore contains another subsequence, say $\{f_{j(r)}\}$ that converges to f υ-a.e.. Therefore, by Scheffé's theorem ([2], 16.11) and by the classical Portmanteau theorem, $\{\mu_{j(r)}\}$ converges weakly to μ. Since weak convergence derives from a (metrizable) topology (see [6] and, for a recent reference, [5] and the literature quoted therein), the sequence $\{\mu_n\}$ converges weakly to μ. ■

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