COMBINATORIAL DESCRIPTION OF PSEUDOSURFACES (*)

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SOMMARIO. - *Viene definita la caratterizzazione di una pseudosuperficie tramite un oggetto matematico discreto: un grafo o più precisamente un grafo bipartito colorato pesato, in maniera biunivoca e modulo isomorfismo (di oggetti matematici discreti) e omeomorfismo (di pseudosuperfici). Una tale corrispondenza è stata considerata per particolari pseudosuperfici da Bayer ed Eisenbud in* [1] *e da Miranda in* [7]. *La loro costruzione è riottenuta come caso particolare* [4].

SUMMARY. - *We want to characterize a pseudosurface by a discrete mathematical object such as a graph or more properly a weighted colored bipartite graph in a* 1-1 *and onto way up to isomorphism (of discrete mathematical objects) and homeomorphism (of pseudosurfaces). Such a correspondence has been considered for particular types of pseudosurfaces by Bayer and Eisenbud in* [1], *and by Miranda in* [7]. *We obtain their construction as a particular case of ours* [4].

1. Preliminaries. To fix our notations about multisets, we give now the definition and some considerations.

DEFINITION 1.1. *Let X be a fixed set and let $N_0 = N \cup \{0\}$ be the set of natural numbers with zero included. Consider functions from X to N_0. Fix such a function*

$$\chi : X \to N_0 \ ;$$

then

$$supp\, \chi = \{x \in X | \chi(x) \neq 0\} \ .$$

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We call $\chi(x)$ *the multiplicity of* x *in* X. *Consider the collection* M *obtained taking each* $x \in supp\ \chi$, $\chi(x)$ *times.* M *in general, is not a set. We call* M *a multiset.* χ *is the characteristic function of* M; *to emphasize this we will write* χ_M. *Put*

$$Card(M) := \sum_{x \in X} \chi_M(x)$$

and

$$Supp(M) := supp\ \chi_M \ .$$

$Supp(M)$ *is an ordinary set. Let* $\lambda \in N_0$ *and let* $x \in X$. *We write* $\lambda x \in M$ *if* $\lambda \leq \chi_M(x)$.

So if $\{a, b, c\}$ is a set, then $\{a, a, b, b, b\}$ is a multiset, but we will, shortly, denote it by $\{2\,a, 3\,b\}$, with the obvious meaning for χ_M. Note that there are five elements in $\{2\,a, 3\,b\}$.

Intuitively a multiset is a set in which we allow repetitions of elements, and the characteristic function χ_M counts how many times an element appears in M.

If χ_M is a $\{0, 1\}$-function, then M in an ordinary set. In this case $Supp(M) = M$.

DEFINITION 1.2. *Let* M_1, M_2 *be multisets. Then*

$$M_1 \subseteq M_2 \text{ iff } Supp(M_1) \subseteq Supp(M_2) \text{ and}$$

$$\chi_{M_1}(x) \leq \chi_{M_2}(x), \text{ for each } x \in M_1 \ .$$

We advise the reader that some elementary prerequisites on graphs are needed. If some term will result unknown, the reader can refer to [5]. But note that the term *subdivision graph* in [5] means a graph obtained from another by a finite sequence of subdivisions, whereas we mean that each edge has been subdivided one time.

2. Pseudosurfaces. We consider the reader acquainted with surfaces as defined in [6]. In particular, if the contrary is not stated, by *surface* we mean a real closed surface, i.e. a compact real 2-manifold without boundary.

By *pseudosurface* we mean a compact topological space $\mathcal{P}$ that fails to be – in a sense to be precised further – a surface at a finite number of points, which are called *singular (points)* and form the subset $Sing(\mathcal{P})$ of $\mathcal{P}$; $sing(\mathcal{P})$ will denote the cardinarlity of $Sing(\mathcal{P})$. Every point in $\mathcal{P}\backslash Sing(\mathcal{P})$ has a neighborhood homeomorphic to the interior of the unit disk; every point $Q \in Sing(\mathcal{P})$ has a neighborhood homeomorphic to a finite number $m_Q(m_Q > 1)$ of disks, all of whose centers have been identified to one point, which in the homeomorphism corresponds to the vertex Q. The number m_Q is constant within all neighborhoods contained in the previous one, and m_Q is called the *multiplicity of Q*. Particular names occur if $m_Q = 2$ or 3: Q is a *node* or *double point* respectively a *triple point.* If $Q \in \mathcal{P}\backslash Sing(\mathcal{P})$ then $m_Q = 1$ and Q is called *non-singular* or *smooth* point.

EXAMPLE 2.1. *A pseudosurface $\mathcal{P}$ can be obtained from a single surface identifying a finite set of points. It can also be obtained by fixing a circle and then shrinking it to a point which thus becomes singular. E.g. the* pseudosphere*: we can obtain it by identifying two points of a sphere or shrinking up a circle – not bounding a disk – of the torus.*

More generally a pseudosurface $\mathcal{P}$ can be viewed as a finite collection of surfaces $\mathcal{V}_1, \ldots, \mathcal{V}_n$ in which finite sets of points are identified [8], [5].

Each surface which contributes to composing the pseudosurface is called a *component* of the pseudosurface. Denote the finite set of components with $Comp(\mathcal{P})$ and its cardinality with $comp(\mathcal{P})$. Note that components are smooth: they have no singular points.

EXAMPLE 2.2. *If $\mathcal{P}$ is a pseudosphere, then the component $\mathcal{V}$ is the sphere.*

To each singular point $Q \in Sing(\mathcal{P})$ there correspond m_Q regular distinct points on $\cup_{j=1}^{comp(\mathcal{P})} \mathcal{V}_j$. Often it is convenient to view $Sing(\mathcal{P})$ as the *support* of a *multiset* $Sing^M(\mathcal{P})$, where we take each $Q \in Sing(\mathcal{P})$ with its multiplicity m_Q. The points that are candidate to form a singular point Q will also be denoted with the letter Q. These points form in each component $\mathcal{V}_j$ a multiset $Sing^{(M)}(\mathcal{V}_j)$ and

$$\begin{aligned} Sing^M(\mathcal{P}) &= \cup_{j=1}^{comp(\mathcal{P})} Sing^M(\mathcal{V}_j) \\ &= \{m_Q Q | Q \in \mathcal{P} \wedge m_Q > 1\} . \end{aligned}$$

As before $sing^M(\mathcal{P})$ and $sing^M(\mathcal{V}_j)$ denote the cardinality of $Sing^M(\mathcal{P})$ and $Sing^M(\mathcal{V}_j)$ respectively. Besides, the names without ^{M} indicate the supports of the relative multisets, respectively the cardinality of the supports. $m_Q(\mathcal{V}_j)$ is the multiplicity of Q on the component $\mathcal{V}_j$.

NOTE 2.3. *In our definition of a pseudosurface we have not excluded the possibility of a component being nonorientable. In the sequel* S_0, S_1, $S_2, \ldots$ *will indicate the orientable surfaces of genus* $0, 1, 2, \ldots$ *and* N_1, $N_2, \ldots$ *the nonorientable surfaces of crosscap number* $1, 2, \ldots$.

While the surfaces are supposed connected we allow for the pseudosurfaces to be disconnected.

Objects considered are triangulable: pseudosurfaces are a particular type of finite 2-complexes, the singularities being placed in the vertices (of the 1-complex of the finite 2-complex). Whittlesey in [9,10,11] has studied invariants and canonical form of finite 2-complexes; the singularities within the present work are Whittlesey's "conical singularities" and his term "node" differs from ours, which is taken from algebraic geometry terminology. Refer to [4], for a more detailed exposition, other examples and further references.

DEFINITION 2.4. *Let* $\gamma(S)$ *be the genus or crosscap number of the surface* S.

We define the species *of* S

$$species(S) := \begin{cases} \gamma(S) & \text{if } S \text{ is orientable} \\ -\gamma(S) & \text{if } S \text{ is nonorientable} \end{cases}$$

the orientability factor

$$\delta_S := \begin{cases} 1 & \text{if } S \text{ is orientable} \\ 0 & \text{if } S \text{ is nonorientable} \end{cases}$$

We recall also that the Euler characteristic of the surface $\mathcal{S}$ is

$$\chi(\mathcal{S}) = \begin{cases} 2 - 2\gamma(\mathcal{S}) & \text{if } \mathcal{S} \text{ is orientable} \\ 2 - \gamma(\mathcal{S}) & \text{if } \mathcal{S} \text{ is nonorientable} . \end{cases}$$

DEFINITION 2.5. *Let u and r be nonnegative integers and let h be an integer.*

A pseudosurface $\mathcal{P}$ is said to be

- u-uniform *or* uniform *if* $m_Q = u$ *for each singular point* $Q \in Sing(\mathcal{P})$,
- r-regular *or* regular *if* $sing^M(\mathcal{V}) = r$ *for each component* $\mathcal{V} \in Comp(\mathcal{P})$,
- h-homogeneous *or* homogeneous *if*

$$species(\mathcal{V}) = h \text{ for each } \mathcal{V} \in Comp(\mathcal{P}) ,$$

i.e. all components have the same Euler characteristic and same orientability factor,

- (u, r, h)-perfect *or* perfect *if is u-uniform, r-regular and h-homogeneous.*

The pseudosurfaces form a category.

The objects of the category are pseudosurfaces. The morphisms can be taken to be all continuous maps which satisfy some extra conditions.

Let $\mathcal{P}$ and $\mathcal{P}'$ be pseudosurfaces.

$$\varphi : \mathcal{P} \to \mathcal{P}'$$

is a *morphism* if

- φ is continuous,
- φ maps components to components,
- $m_Q(\mathcal{V}) \leq m_{\varphi(Q)}(\varphi(\mathcal{V}))$, for each $\mathcal{V} \in Comp(\mathcal{P})$, for each $Q \in \mathcal{P}$.

A morphism maps singular points into singular points. The category of pseudosurfaces will be denoted by $PseudoS$. A morphism φ is an *isomorphism* if and only if φ is a homeomorphism; moreover

$$m_Q(\mathcal{V}) = m_{\varphi(Q)}(\varphi(\mathcal{V})) ,$$

for each $\mathcal{V} \in Comp(\mathcal{P})$, for each $Q \in \mathcal{P}$. This implies that $m_Q \leq m_{\varphi(Q)}$, but since the same happens for φ^{-1} then

$$m_Q = m_{\varphi(Q)} .$$

A slightly different category of pseudosurfaces, $PseudoS^w$, can be defined requiring that morphisms preserve the species of the components i.e.

$$species(\varphi(\mathcal{V})) = species(\mathcal{V}) \text{ for each } \mathcal{V} \in Comp(\mathcal{P}) .$$

The isomorphisms are in reality the same in both categories.

DEFINITION 2.6. *A pseudosurface $\mathcal{P}$ is called*

- singular transitive *if for each $Q, Q' \in Sing(\mathcal{P})$ there exists an automorphism φ such that $\varphi(Q') = Q$,*
- regular transitive *if for each $j, k, 1 \leq k, j \leq comp(\mathcal{P})$, there exists* an automorphism *φ that maps $Sing^M(\mathcal{V}_j)$* to *$Sing^M(\mathcal{V}_k)$,*
- homogeneous transitive *if for each $j, k, 1 \leq k, j \leq comp(\mathcal{P})$, there exists an automorphism φ that maps $\mathcal{V}_j$* to *$\mathcal{V}_k$,*
- perfect transitive *if it is singular, regular and homogeneous transitive.*

PROPOSITION 2.7. *If $\mathcal{P}$ is singular (regular, homogeneous) transitive, then it is u-uniform (r-regular, h-homogeneous) for some $u(r, h)$.*

Proof. Easy, recalling definitions. ■

The reverse implication of proposition 2.7 is not true as can be shown by simple counterexamples.

3. Related combinatorial objects. In this section we establish three equivalent combinatorial tools for describing pseudosurfaces: weighted generalized hypergraphs, weighted nonnegative integer matrices and weighted colored bipartite graphs.

Let begin with some considerations on graphs.

From the combinatorial viewpoint a *graph* $\mathcal{G}$ consists of a set $V(\mathcal{G})$ of *vertices* and a set $E(\mathcal{G})$ of *edges*. Each edge $e \in E(\mathcal{G})$ has an *endpoint* *set* ∂e containing either one or two elements of the vertex set $V(\mathcal{G})$. The multiset (of subsets)

$$I(\mathcal{G}) = \{\partial e | e \in E(\mathcal{G})\}$$

is the *incidence structure* of $\mathcal{G}$. Let $v_{\mathcal{G}}$ denote $Card(V(\mathcal{G}))$ and let $e_{\mathcal{G}}$ denote $Card(E(\mathcal{G}))$. The incidence matrix $\mathcal{I}(\mathcal{G})$ has $v_{\mathcal{G}}$ rows and $e_{\mathcal{G}}$ columns. The entry $i(v, e)$ is defined as follows:

$$i(v,e) = \begin{cases} 2 & \text{if } \partial e = \{v\}, \\ 1 & \text{if } v \in \partial e, \text{ and } Card(\partial e) = 2 \\ 0 & \text{otherwise.} \end{cases}$$

But ∂e can be viewed as the support of a *multiset* ∂e^M that contains in all cases both endpoints of e; so if e is a loop based on v then $\partial e^M = \{v, v\} = \{2v\}$. With this notation $i(v, e)$ is the multiplicity of v in ∂e^M.

Also we can think about an edge e as a map

$$F_e : V(\mathcal{G}) \to \{0, 1, 2\},$$

such that putting

$$\partial e^M = \{F_e(v) \cdot v | F_e(v) \neq 0\},$$

it results $Card(\partial e^M) = 2$ for every edge e of $\mathcal{G}$. Then we interpret $i(v, e)$ as $F_e(v)$. F_e is the characteristic map of ∂e^M.

From this point of view we can see how we may generalize the concept of graph.

A *generalized hypergraph* $\mathcal{H}$ is a set $V(\mathcal{H})$ of *(hyper)vertices* and a set $E(\mathcal{H})$ of *(hyper)edges*, each assigned by an element of $Map(V(\mathcal{H}), N_{\mathcal{H}})$, where $N_{\mathcal{H}} \subseteq N_0$.

As before each edge $e \in E(\mathcal{H})$ has an *endpoint multiset* ∂e^M; it is the multiset

$$\partial e^M = \{F_e(v) \cdot v | F_e(v) \neq 0\},$$

and an *endpoint set* $\partial e = Supp(\partial e^M)$.

The multiset $I(\mathcal{H}) = \{\partial e^M | e \in E(\mathcal{H})\}$ is the *incidence structure* of $\mathcal{H}$.

We will deal only with finite generalized hypergraphs, so the sets $V(\mathcal{H})$ and $E(\mathcal{H})$ will be finite.

If $V(\mathcal{H})$ is finite then for each edge $e \in E(\mathcal{H})$, $Supp(F_e)$ is finite and also $0 \leq Card(\partial e^M) < \infty$. If $Card(\partial e) = 0,\ 1, \geq 2$ then e is called an *isolated edge, pre-edge* or *effective edge* respectively. If all edges are effective then the generalized hypergraph will be called *effective* too. The set of effective edges is denoted by $E^{FF}(\mathcal{H})$.

To a generalized hypergraph $\mathcal{H}$ we associate a matrix $\mathcal{I} = \mathcal{I}(\mathcal{H})$ with $v_{\mathcal{H}}$ rows and $e_{\mathcal{H}}$ columns. For each vertex v and each edge e put

$$i(v,e) := F_e(v) .$$

This matrix is called the *incidence matrix* of the generalized hypergraph. A row represents a vertex v and a column an edge e. The entries $i(v,e)$ are non-negative integers. The sum of the entries of the column corresponding to the edge e, is equal to $Card(\partial e^M)$. The sum of the entries of a row is called the *valence* of the corresponding vertex.

We will see later that putting conditions on ∂e or on $N_{\mathcal{H}}$ we can obtain the objects which are generalized by the new definition.

A *generalized hypergraph map* $h : \mathcal{H} \to \mathcal{H}'$ consist of a vertex function

$$h_V : V(\mathcal{H}) \to V(\mathcal{H}')$$

and an edge function

$$h_E : E(\mathcal{H}) \to E(\mathcal{H}')$$

such that it is compatible with the incidence structure i.e.

$$F_{h_E(e)}(h_V(v)) \geq F_e(v),\ \forall v \in V(\mathcal{H}, \forall e \in E(\mathcal{H}) .$$

For the incidence matrix $\mathcal{I}(\mathcal{H})$ this is stated by

$$i(h_V(v), h_E(e)) \geq i(v,e),\ \forall v \in V(\mathcal{H}),\ \forall e \in E(\mathcal{H}) .$$

h is an *isomorphism* if both h_V and h_E are bijections and for each pair v, e we have

$$F_{h_E(e)}(h_V(v)) = F_e(v)$$

or respectively

$$i(h_V(v), h_E(e)) = i(v, e) \ .$$

Generalized hypergraphs and their maps form a category that we denote by $HypGraph$.

Let see how generalized hypergraphs are the same structure as nonnegative rectangular matrices, and subsequently how to obtain a bipartite colored graph.

Let $\mathcal{M}$ be a matrix. Let $R(\mathcal{M})$ the set of row indices and let $C(\mathcal{M})$ denote the column indices of $\mathcal{M}$. Now we can define *matrix maps* as follows.

A *matrix map* $h : \mathcal{M} \to \mathcal{M}'$ consists of a *row function*

$$h_r : R(\mathcal{M}) \to R(\mathcal{M}')$$

and a *column function*

$$h_c : C(\mathcal{M}) \to C(\mathcal{M}')$$

such that it is compatible with the incidence structure:

$$(\mathcal{M}')_{h_R(v),h_C(e)} \geq (\mathcal{M})_{v,e}, \forall v \in R(\mathcal{M}),\ \forall e \in C(\mathcal{M}) \ .$$

The nonnegative integer matrices with matrix maps obviously form a category. $NatMat$ that is equivalent to the category $HypGraph$.

Let $\mathcal{M}$ be an arbitrary nonnegative matrix. We will form another matrix $\mathcal{A}^V(\mathcal{M})$ as follows.

$$\mathcal{A}^V(\mathcal{M}) := \begin{pmatrix} 0 & \mathcal{M} \\ \mathcal{M}^t & 0 \end{pmatrix} .$$

Clearly $\mathcal{A}^V(\mathcal{M})$ is a square nonnegative symmetric matrix. Therefore it can be interpreted as an adjacency matrix of a bipartite (multi)-graph $\mathcal{B} = \mathcal{B}(\mathcal{M})$. Moreover, we may color the vertices of $\mathcal{B}$ black and white in such a way that the black vertices correspond to the rows of $\mathcal{M}$ and that the white ones correspond to the columns of $\mathcal{M}$. Hence we may take $R(\mathcal{M})$ for the black and $C(\mathcal{M})$ for the white partite set of $\mathcal{B}$. So $\mathcal{B}$ is a *colored* bipartite graph. Now we form a category that has finite colored bipartite graphs as objects. For morphisms we take ordinary (dimension-preserving) graph maps that preserve colors and are compatible with the adjacency structure:

$$Card(E\langle h(v_b), h(v_w)\rangle) \geq Card(E\langle v_b, v_w\rangle)$$

for each black vertex v_b and for each white vertex v_w, where $E\langle x, y\rangle$ is the subgraph induced on the vertices x and y.

This yields a category *CoBip* which is equivalent to HypGraph and to *NatMap*. From now on we will speak of the same structure which can be regarded from three different viewpoints. The three objects $\mathcal{H}, \mathcal{M}$, and $\mathcal{B}$ will be called *related*.

The objects of *HypGraph* are very general so we often impose other conditions. Here we list a few such conditions.

a) For each $e \in E(\mathcal{H})$, ∂e^M is not the void set.
b) There are at least two elements in ∂e^M for each $e \in E(\mathcal{H})$.
c) There are at most two elements in ∂e^M for each $e \in E(\mathcal{H})$.
d) ∂e^M is an ordinary set for each $e \in E(\mathcal{H})$.
e) There are exactly two elements in ∂e^M for each $e \in E(\mathcal{H})$.
f) $I(\mathcal{H})$ is an ordinary set.
g) Each element of $V(\mathcal{H})$ appears in ∂e for at least one $e \in E(\mathcal{H})$.

If a) does not hold then $\mathcal{H}$ contains the so-called *free edges*. If b) is satisfied $\mathcal{H}$ is an effective generalized hypergraph. If b) is not satisfied then $\mathcal{H}$ may contain the so-called *pre-edges*. And if c) is satisfied, but e) is not, then $\mathcal{H}$ is a pregraph. If a) and d) are satisfied then $\mathcal{H}$ is a *hypergraph* [2]. If e) is satisfied then $\mathcal{H}$ is an ordinary graph, possibly with loops. In such a case $\mathcal{I} = \mathcal{I}(\mathcal{H})$ is the ordinary incidence matrix associated with the graph, and the sum of the entries of a column is always 2; the condition implies $\mathrm{N}_{\mathcal{H}} = \{0, 1, 2\}$; if $Card(\partial e) = 1$ then e is a loop, if $Card(\partial e) = 2$ then e is a proper edge. If in addition f) holds then $\mathcal{H}$ has no parallel edges. If d) and e) are satisfied then $\mathcal{H}$ is a graph with no loops. If g) is satisfied there are no isolated vertices.

The conditions above can be rephrased in terms of integer nonnegative matrices $\mathcal{M}$:

a) No column of $\mathcal{M}$ contains only zeros.
b) The column sums in $\mathcal{M}$ are all at least 2.
c) The column sums in $\mathcal{M}$ are all at most 2.
d) $\mathcal{M}$ is a zero-one matrix.
e) The column sums in $\mathcal{M}$ are all equal to 2.
f) All columns of $\mathcal{M}$ are pairwise distinct.
g) No row of $\mathcal{M}$ contains only zeros.

and in terms of colored bipartite graphs $\mathcal{B}$:

a) No white vertex of $\mathcal{B}$ is isolated.
b) The valence of each white vertex is at least 2.
c) The valence of each white vertex is at most 2.
d) $\mathcal{B}$ has no parallel edges.
e) $\mathcal{B}$ is a subdivision graph $S(\mathcal{G})$ of some graph $\mathcal{G}$ on the black vertices.
f) Any two white vertices of $\mathcal{B}$ have distinct adjacency degrees.
g) No black vertex is isolated.

If e) holds we will call the related objects *graphical.*

From now on we will assume that the condition b) holds, unless stated otherwise.

Since $\mathcal{B}$ is a graph we can test if it is connected or not. Using graph connectivity it is then possible to define *connectivity* for $\mathcal{H}$ and for $\mathcal{M}$.

To each vertex v of the generalized hypergraph, or equivalently to each row index of $\mathcal{M}$, or to each black vertex of $\mathcal{B}$ we will associate an integer $\varpi(v) \in Z$, that we will call the *weight* of v. We obtain a column matrix $\varpi = \varpi(\mathcal{X}), \mathcal{X} = \mathcal{H}, \mathcal{B}, \mathcal{M}$, which we append to $\mathcal{X}$. The resulting object $(\mathcal{X}, \varpi(\mathcal{X}))$ will be called a *weighted generalized hypergraph*, respectively *weighted nonnegative matrix* or *weighted colored bipartite graph.*

Weighted objects also form categories. One has to be careful with morphisms. In this case the weights have to be preserved.

The relative categories are denoted by $HypGraph^{\varpi}$, $NatMat^{\varpi}$ and $CoBip^{\varpi}$.

For a weighted generalized hypergraph we put the following:

Let u and r be nonnegative integers and h a constant in the set of weights.

DEFINITION 3.1. *A weighted generalized hypergraph $\mathcal{H}$ will be called:*

- u-uniform *or* uniform *if* $Card(\partial e^M) = u$ *for each edge* e,
- r-regular *or* regular *if the valence of each vertex equals* r,
- h-homogeneous *or* homogeneous *if* $\varpi(v) = h$ *for each vertex* $v \in V(\mathcal{H})$,
- *(u,r,h)*-perfect *if is u-uniform, r-regular and h-homogeneous.*

We state analogous definitions for weighted nonnegative matrix and weighted colored bipartite graphs. Let u, r and h be as before.

DEFINITION 3.2. *A weighted nonnegative matrix $\mathcal{M}$ will be called:*

- u-uniform *or* uniform *if the column sum is equal to u for each column,*
- r-regular *or* regular *if the row sum is equal to r for each row,*
- h-homogeneous *or* homogeneous *if* $weight(j) = h$ *for each row* $j \in R(\mathcal{M})$,
- (u, r, h)-perfect *if is u-uniform, r-regular and h-homogeneous.*

DEFINITION 3.3. *A weighted colored bipartite graph $\mathcal{B}$ will be called:*

- u-uniform *or* uniform *if the valence of each white vertex is equal to* u,
- r-regular *or* regular *if the valence of each black vertex is equal to r,*
- h-homogeneous *or* homogeneous *if the weight of each black vertex is equal to h,*
- *(u,r,h)*-perfect *if is u-uniform, r-regular and h-homogeneous.*

NOTE 3.4. *If the weighted generalized hypergraph is in reality a graph we will call it a weighted graph. We remark that some authors use the term weighted graph for a graph labelled on edges. In* [5] *the terms voltage graph and current graph are used for directed (oriented) graphs labelled on edges. We choose the expression weighted graph for those labelled on vertices.*

PROPOSITION 3.5. *Let $\mathcal{H}$ be a hypergraph and $\mathcal{B}$ be a bipartite graph. Let $\mathcal{H}$ and $\mathcal{B}$ be related graphical objects. Then $\mathcal{H}$ is a graph and its subdivision graph $S(\mathcal{H})$ is isomorphic to $\mathcal{B}$.*

Proof. Let $e \in E(\mathcal{H})$. There are exactly two elements in ∂e. So $Card(\partial e) = 2$ and $0 \leq F_e(v) \leq 2$ for each vertex v. That implies $N_{\mathcal{H}} = \{0,1,2\}$. Then $\mathcal{H}$ is a graph.

Let $S(\mathcal{H})$ be its subdivision graph: call black the vertices of $\mathcal{H}$ in $S(\mathcal{H})$, and white the subdivision vertices.

The black vertices of $\mathcal{B}$ are the same as the black vertices of $S(\mathcal{H})$, as they are the vertices of $\mathcal{H}$.

We want to define an isomorphism ψ between $S(\mathcal{H})$ and $\mathcal{B}$. Let ψ be the identity on the black vertices. Take an edge $e \in E(\mathcal{H})$. To e there corresponds on $S(\mathcal{H})$ a subdivision vertex $s(e)$ and on $\mathcal{B}$ a white vertex $b^{w}(e)$. Let

$$\psi(s(e)) = b^{w}(e) .$$

Let e be a proper edge. $\partial e = \{h_1, h_2\}$. Then e generates on $S(\mathcal{H})$ two edges f_1, f_2 that are not parallel: $\partial f_i = \{h_i, s(e)\}$, $i = 1,2$.

Let g_i be the edge on $\mathcal{B}$ such that $\partial g_i = \{h_i, b^{w}(e)\}$, $i = 1,2$.

Define $\psi(f_i) = g_i$.

If e is a loop it generates also in this case two edges f_1, f_2, but they are parallel. $i(h,e) = 2$ then in $\mathcal{B}$, $b^{w}(e)$ is connected to h with two parallel edges g_1, g_2. Put $\psi(f_i) = g_i$. In effect there is no matter which goes to which, the only requirement is that ψ results injective.

Then ψ so defined is a graph map, is bijective and so is a graph isomorphism. ■

4. Combinatorial description of pseudosurfaces. A pseudosurface $\mathcal{P}$ has discrete elements that characterize it: the disjoint components elements of $Comp(\mathcal{P})$, their Euler characteristics and orientability factor, the points in $Sing(\mathcal{P})$.

Take a 2-uniform pseudosurface with orientable components: it is the topological model of a complex algebraic projective curve with only nodes. In [1] and [7] to each complex algebraic curve with only nodes, it is associated a weighted graph: vertices come from components and edges from nodes.

We want to generalize this concept: the way is to associate to a pseudosurface a weighted generalized hypergraph, such that to each component it is associated a vertex and to each singular point an edge. For simplicity we will use the same letter for components and vertices, and for singular points and edges: capital if we are considering elements of the pseudosurface, small if we are considering elements of the generalized hypergraph.

Let $\mathcal{P}$ be the pseudosurface. Consider the generalized hypergraph $\mathcal{H} = \mathcal{H}(\mathcal{P})$ obtained taking a vertex v for each component $\mathcal{V} \in Comp(\mathcal{P})$ and an edge q for each singular point $Q \in Sing(\mathcal{P})$, where

$$F_q(\cdot) = m_Q(\cdot) ,$$

obviously on the left the argument is a vertex, and on the right is the corresponding component.

$\mathbf{N}_{\mathcal{H}} = \{0, 1, \ldots, m\}$, $m = \max_{Q \in Sing(\mathcal{P}), \mathcal{V} \in Comp(\mathcal{P})} m_Q(\mathcal{V})$. The weights are determined by the species: $\varpi(v) = species(\mathcal{V})$.

The vertex set $V(\mathcal{H})$ is in bijective correspondence with $Comp(\mathcal{P})$ and the edge set $E(\mathcal{H})$ with $Sing(\mathcal{P})$.

The entries of the incidence matrix $\mathcal{I}(\mathcal{H})$ are denoted by $i(v, q)$.

Let $v_{\mathcal{H}}$ denote $Card(V(\mathcal{H}))$ and let $e_{\mathcal{H}}$ denote $Card(E(\mathcal{H}))$. Let n_Q denote the number of different components that Q belongs to. Recall that m_Q denotes the multiplicity of Q. We have the following equalities.

$$\begin{aligned} m_Q &= \sum_{v \in V(\mathcal{H})} F_q(v) \\ &= \sum_{v \in V(\mathcal{H})} i(v, q) \end{aligned} \tag{1}$$

$$v_{\mathcal{H}} = comp(\mathcal{P}) \tag{2}$$

$$e_{\mathcal{H}} = \sum_{q \in E(\mathcal{H})} 1 = sing(\mathcal{P}) \tag{3}$$

$$sing^M(\mathcal{V}) = \sum_{q \in E(\mathcal{H})} i(v, q) \tag{4}$$

$$\begin{aligned} n_Q &:= m_Q - \sum_{v \in V(\mathcal{H}), i(v,q) \neq 0} i(v, q) - 1) \\ &= \sum_{v \in V(\mathcal{H}), i(v,q) \neq 0} 1 \\ &= Card(\{v | i(v, q) \neq 0\}) \end{aligned} \tag{5}$$

Given a weighted generalized hypergraph **H**, with $\mathcal{I}(\mathcal{H})$ and $\varpi(\mathcal{H})$, we can recover a pseudosurface $\mathcal{P} = \mathcal{P}(\mathcal{H})$, as we know the number of components, their species and how to match together points to obtain the singular ones.

$Card(\partial q^M)$ is at least 2 for each edge, so the edges really correspond to singular points.

If the valence of each vertex is at least 1, then on each component there is at least one singular point.

If we admit for a while that the weighted generalized hypergraph is not effective then $Card(\partial q^M)$ can be also 0 or 1. If $Card(\partial q^M) = 1$ and let $\mathcal{V}$ be such that $F_q(v) = 1$, then this means on $\mathcal{V}$ we fix a smooth point Q which is considered singular. $Card(\partial q^M) = 0$ is a degenerate case as no component can be concerned, so it fix no point on the pseudosurface, but an isolated point. In any case points Q, corresponding to edges q with $Card(\partial q^M) = 0$ or 1 are not to be considered in $Sing(\mathcal{P})$ or $Sing(\mathcal{V})$.

If $\mathcal{H}$ is connected then the resulting object is also connected.

The following proposition generalizes the result of Miranda [7], where this was proved for graphical pseudosurfaces P such that $\delta_{\mathcal{V}} = 1$ for each $\mathcal{V} \in Comp(\mathcal{P})$.

PROPOSITION 4.1. *Two pseudosurfaces are homeomorphic if and only if the corresponding weighted generalized hypergraphs are isomorphic.*

Proof. Let $\mathcal{P}$ and $\mathcal{P}'$ be two homeomorphic pseudosurfaces. So they have the same number of components, with same orientability factor and same genus or crosscap number and singularities must be obtained identifying sets of points in an analogous way. Then $\mathcal{H} = \mathcal{H}(\mathcal{P})$ and $\mathcal{H}' = \mathcal{H}(\mathcal{P}')$ are isomorphic as they have the same number of vertices, with the same weights, and edges connect vertices with same incidence structure.

Conversely if $\mathcal{H}$ and $\mathcal{H}'$ are two isomorphic effective weighted generalized hypergraphs, $\mathcal{P} = \mathcal{P}(\mathcal{H})$ and $\mathcal{P}' = \mathcal{P}(\mathcal{H}')$ have the same number of components and the identification of points take place similarly. Next corresponding components have the same Euler characteristic and orientability factor so are homeomorphic. So $\mathcal{P}$ and $\mathcal{P}'$ are homeomorphic. ∎

PROPOSITION 4.2. *The categories $PseudoS^{\varpi}$ and $HypGraph^{\varpi}$ are equivalent*

Proof. We have to define two covariant functors G_1 and G_2

$$G_1 : PseudoS^{\varpi} \to HypGraph^{\varpi}$$

$$G_2 : HypGraph^{\varpi} \to PseudoS^{\varpi}$$

which will satisfy the conditions of an equivalence, see [3].

Let G_1 on objects be defined as established before

$$G_1(\mathcal{P}) := \mathcal{H}(\mathcal{P}) \ .$$

If we have a morphism

$$\varphi : \mathcal{P} \to \mathcal{P}'$$

then we define

$$G_1(\varphi) : \mathcal{H}(\mathcal{P}) \to \mathcal{H}(\mathcal{P}')$$

to be the morphism of weighted generalized hypergraphs such that

$$G_1(\varphi)_V(v) = v' \text{ if } \varphi(\mathcal{V}) = \mathcal{V}'$$

and

$$G_1(\varphi)_E(q) = q' \text{ if } \varphi(Q) = Q' \ .$$

$G_1(\varphi)$ is compatible with the incidence structure as φ is required to be compatible with multiplicities:

$$F_q(v) = m_Q(\mathcal{V} \leq m_{\varphi(Q)}(\varphi(\mathcal{V})) = F_{G_1(\varphi)_E(q)}(G_1(\varphi)_V(v)) \ .$$

$G_1(\varphi)$ preserves weights as φ preserves species.

Let G_2 on objects be defined as follows

$$G_2(\mathcal{H}) := \mathcal{P}(\mathcal{H}) \ .$$

Let

$$h : \mathcal{H} \to \mathcal{H}'$$

be a morphism of $HypGraph^{\varpi}$. If $h(v) = v'$ then $\varpi(h(v)) = \varpi(v)$. The corresponding components $\mathcal{V}$ and $\mathcal{V}'$ have the same species. It follows that $\mathcal{V}$ and $\mathcal{V}'$ are homeomorphic. Here is where the edge function h_E comes in:

$$G_2(h)(Q) = Q' \text{ if } h_E(q) = q' .$$

Choose a homeomorphism such that the image of $Sing^M(\mathcal{V})$ is contained in $Sing^M(\mathcal{V}')$. The components intersect only on the singular points, where the images are determined by h_E.

Moreover, the conditions on the multiplicities are verified, as h is compatible with the incidence structure.

The homeomorphisms on each component glue together to form a morphism

$$G_2(h) : \mathcal{P}(\mathcal{H}) \rightarrow \mathcal{P}(\mathcal{H}') .$$

We consider further the functors

$$G_1 \circ G_2 : HypGraph^{\varpi} \rightarrow HypGraph^{\varpi}$$

and

$$G_2 \circ G_1 : PseudoS^{\varpi} \rightarrow PseudoS^{\varpi} .$$

From the previous proposition we can find two functorial isomorphisms f and f' that make commutative the diagrams of the following form

$$\begin{array}{ccc} \mathcal{H} & \stackrel{f(\mathcal{H})}{\rightarrow} & G_1(G_2(\mathcal{H})) \\ h\downarrow & & \downarrow G_1(G_2(h)) \\ \mathcal{H}' & \stackrel{f(\mathcal{H}')}{\rightarrow} & G_1(G_2(\mathcal{H}')) \\ \mathcal{P} & \stackrel{f'(\mathcal{P})}{\rightarrow} & G_2(G_1(\mathcal{P})) \\ \varphi\downarrow & & \downarrow G_2(G_1(\varphi)) \\ \mathcal{P}' & \stackrel{f'(\mathcal{P}')}{\rightarrow} & G_2(G_1(\mathcal{P}')) \end{array}$$

and $G_2 f = f' G_2$.

So the categories $HypGraph^{\varpi}$ and $PseudoS^{\varpi}$ are equivalent. ∎

PROPOSITION 4.3. *Let* $\mathcal{P} = \mathcal{P}(\mathcal{H})$. $\mathcal{H}$ *is a weighted graph, possibly with loops if and only if all the singularities on* $\mathcal{P}$ *are nodes, i.e. if* $m_Q = 2$ *for all* $Q \in Sing(\mathcal{P})$ *i.e. if* $\mathcal{P}$ *is 2-uniform.*

Proof. The proof is trivial. Observe that all the entries of the incidence matrix of $\mathcal{H}$ are non negative. The sum of the entries of a column give the multiplicity of a singular point (1).

So, suppose $\mathcal{H}$ is a graph. Then the sum of the entries of a column is always 2 and the singular points are nodes. Conversely, suppose all the singular points are nodes, then the sum of the entries of a column has to be 2. This can happen only if all entries are zero except one that is equal 2 or except two that are equal 1. ■

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