THE SPECTRUM OF THE LAPLACE OPERATOR FOR THE MANIFOLD $S^{2n-1} \times S^{2m-1}/\Gamma$ (*)

by GR. TSAGAS (in Thessaloniki) (**)

SOMMARIO. - *Sia* (M, g) *una varietà Riemanniana orientabile compatta. Sia* Γ *un sottogruppo finito di isometrie di* $I(M)$ *come in seguito caratterizzato. Scopo del presente lavoro è di determinare lo spettro della varietà* $S^{2n-1} \times S^{2m-1}/\Gamma$.

SUMMARY. - *Let* (M, g) *be a compact orientable Riemannian manifold. Let* Γ *be a finite subgroup of isometries of* $I(M)$ *which acts fixed point freely. The aim of the present paper is to determine the spectrum of the manifold* $S^{2n-1} \times S^{2m-1}/\Gamma$.

1. Introduction. Let (M, g) be a compact orientable Riemannian manifold. Let Γ be a finite subgroup of the group of isometries $I(M)$ which acts fixed point freely on M. One of the problems of the spectrum is to determine $Sp(M/\Gamma)$.

This problem has been solved, if (M, g) is the unit sphere (S^n, g_0), where g_0 is the standard Riemannian metric with constant sectional curvature 1 and Γ a finite fixed point free subgroup of $0(n+1)$ [3].

The aim of the present paper is to solve the same problem, when (M, g) has the form $(S^n \times S^m,\ g_0 \times g'_0)$.

The whole paper contains four paragraphs.

The second paragraph deals with some general properties of the $Sp(M \times N/\Gamma,\ g \times h/\Gamma)$, where $\Gamma \subset I(M \times N)$ is finite group.

The $Sp(S^n \times S^m/\Gamma$, when $\Gamma = \mathbf{Z}_q$, is computed in the third paragraph.

The last paragraph includes explicitly the estimates of $Sp(S^n \times S^m/\Gamma)$, when $\Gamma = \mathbf{Z}_3$.

(*) Pervenuto in Redazione il 9 febbraio 1989.

(**) Indirizzo dell'Autore: Mathematics Division – School of Technology – Aristotle Univesity of Thessaloniki – 54006 Thessaloniki (Greece).

2. Let S^n be the unit sphere centered at the origin in the $(n+1)$-dimensional Euclidean space $\mathbf{R}^{n+1}$. We denote by $O(n+1)$ the orthogonal group acting on $\mathbf{R}^{n+1}$.

A finite subgroup Γ of $O(n+1)$ is said to be a fixed point free subgroup of $O(n+1)$, if for any $\gamma \in \Gamma$ with $\gamma \neq I_{n+1}$ (the unit matrix in $O(n+1)$) 1 is not an eigenvalue.

Now, we consider the product manifold $S^{2n-1} \times S^{2m-1}$ whose group of isometries is $O(2n) \times O(2m)$. Let Γ be a finite cyclic subgroup of order q of $O(2n) \times O(2m)$, which acts fixed point freely on the manifold $(S^{2n-1} \times S^{2m-1}, g_0 \times g_0')$.

Let γ be a generator of the group Γ. We assume that the eigenvalues of γ are of the form

$$\delta^{p_1}, \dots, \delta^{p_n}, \delta^{p_{n+1}}, \dots, \delta^{p_{n+m}}, \delta^{-p_1}, \dots, \delta^{-p_n}, \delta^{-p_{n+1}}, \dots, \delta^{-p_{n+m}}$$

where

$$\delta = \exp \frac{2\pi\sqrt{-1}}{q}$$

and the positive integers $p_1, \dots, p_n, p_{n+1}, \dots, p_{n+m}$ are prime to q.

The cyclic group $\Gamma = \{\gamma^l\}_{l=0}^{q-1}$ acts fixed point freely on $S^{2n-1} \times S^{2m-1}$.

In order to study this problem we obtain some results for the general case. Let $(M, g), (N, h)$ be two compact and orientable Riemannian manifolds. From these manifolds we take the manifold $(M \times N, g \times h)$ and consider the diagram

$$\begin{array}{ccccc} & & (M \times N,\ g \times h) & & \\ & p \swarrow & & \pi \searrow & \\ (M, g) & & & & (N, h) \\ & f \searrow & & \swarrow \varphi & \\ & & \mathbf{R} & & \end{array}$$

If f and φ are two functions on M and N respectively, then $f \circ p$ and $\varphi \circ \pi$ are functions on $M \times N$. We denote by $\Delta^{M \times N}$, Δ^M and Δ^N the Laplace operators on the Riemannian manifolds $(M \times N,\ g \times H)$, (M, g) and (N, h) respectively. It can easily be proved

$$\Delta^{M \times N}[(f \circ p) \times (\varphi \circ \pi)] = (\varphi \circ \pi) \times [\Delta^M(f) \circ p] + (f \circ p) \times [\Delta^N(\varphi) \circ \pi]. \tag{2.1}$$

If f is an eigenfunction of Δ^M with eigenvalue λ and φ is an eigenfunction of Δ^N with eigenvalue μ, then the relation (2.1) takes the form

$$\Delta^{M\times N}[(f\circ p)\times(\varphi\circ\pi)] = (\lambda+\mu)[(f\circ p)\times(\varphi\circ\pi)]\ , \qquad (2.2)$$

that means $(f\circ p)\times(\varphi\circ\pi)$ is an eigenfunction for the Laplace operator $\Delta^{M\times N}$ with eigenvalue $\lambda+\mu$.

Let (M',g') be a compact and orientable Riemannian manifold. We denote by $C^\infty(M')$ the space of complex valued C^∞-functions on M'. From M' we obtain

$$Sp(M',g') = \{\lambda/\Delta^{M'}f = \lambda f, f\in C^\infty(M'),\ \lambda\in\mathbf{R}\}\ .$$

We consider the subspace $Q(M',g')$ of $C^\infty(M')$ defined as follows

$$Q(M',g') = \sum_{\lambda\in Sp(M',g')} Q_\lambda(M',g')\ ,$$

where $Q_\lambda(M',g')$ is the vector space consisting of eigenfunctions with eigenvalue λ. We also denote this space by $Q^\lambda(M',g')$, which is called λ-eigenspace of (M',g') and is finite dimension. The subset $Q(M',g')$ is dense in $C^\infty(M')$ provided with the topology defined by the inner product $\langle f_1,f_2\rangle = \int_{M'} f_1 f_2\, dM'$, where dM' is the volume element on M'.

We consider the subspace of $C^\infty(M\times N)$ generated by all products of the form $(f\circ p)\times(\rho\circ\pi)$, where $f\in Q(M,g)$ and $\varphi\in Q(N,h)$. This vector subspace of $C^\infty(M\times N)$ is denoted by $p^*Q(M,g)\circ\pi^*Q(N,h)$. This is isomorphic onto the vector space $Q(M,g)\otimes Q(N,h)$. The following relations hold

(i) $p^*Q(M,g)\otimes\pi^*Q(W,h) = Q(M\times N,\ g\times h)$ (2.3)

(ii) $Sp(M\times N,\ g\times h) = \{\lambda+\mu/\lambda\in Sp(M,g), \mu\in Sp(N,h)\}$ (2.4)

(iii) $Q_v(M\times N, g\times h) = \Sigma p^*Q_\lambda(M,g)\otimes\pi^*Q_\mu(N,h)$ (2.5)

$$\lambda\in Sp(M,g)$$
$$\mu\in Sp(N,h)$$
$$v = \lambda+\mu\in Sp(M\times N,\ g\times h)\ .$$

THEOREM 2.1. *Let (M,g), (N,g) be two compact and orientable Riemannian manifolds. Let Γ be a finite subgroup of $I(M\times N)$, where*

$I(M \times N)$ is the group of isometries on $(M \times N,\ g \times h)$, which acts fixed point freely and preserves the separate coordinate system on the product manifold $M \times N$. Then the eigenfunctions on the manifold $(M \times N/\Gamma,\ g \times h/\Gamma)$ are of the forms $(f \circ p) \times (\varphi \circ \pi)$, where f and φ are the eigenfunctions on the manifolds (M, g) and (N, h) respectively, which have the property $\Gamma(f) = tf$ and $\Gamma(\varphi) = s\varphi$ where $t,\ s \in \mathbf{C}$ and $ts = 1$.

Proof. Let $(V_i, \phi_i)_{i\in I}$ and $(V_j, \psi_j)_{j\in J}$ be two atlas on $M(g)$ and (N, h) respectively. Then $(V_i \times V_j,\ \phi_i \times \psi_j)_{(i,j)\in I\times J)}$ is also an atlas on $(M \times N,\ g \times h)$. Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ be two local coordinate systems on the charts (V_i, ϕ_i) and (V_j, ψ_j) respectively, where $\dim M = n$ and $\dim N = m$. Then $(x_1, \ldots, x_n;\ y_1, \ldots, y_m)$ is a local coordinate system on the chart $(V_i \times V_j, \phi_j \times \psi_j)$. If γ is an element of the group Γ, then

$$\begin{aligned} \gamma : M \times N \to M \times N,\ \gamma : V_i \times V_j \to& \\ \to V_{i'} \times V_{j'}, \gamma : (x_1, \ldots, x_n;\ y_1, \ldots, y_m) \to& \\ \to (\gamma(x_1, \ldots, x_n), \gamma(y_1, \ldots, y_m))& \\ = (x'_1, \ldots, x'_n;\ y'_1, \ldots, y'_m)& \end{aligned}$$

which means that it preserves the separate coordinate systems.

If $f \in Q_\lambda(M, g)$ and $\varphi \in Q_\mu(N, h)$, then we obtain

$$\Delta^{M\times N}[(f \circ p) \times (\varphi \circ \pi)] = (\lambda + \mu)[(f \circ p) \times (\varphi \circ \pi)]\ . \qquad (2.6)$$

We assume that the eigenfunctions f and φ have the property

$$\Gamma(f) = tf,\ \Gamma(\varphi) = s\varphi,\ t,\ s \in \mathbf{C},\ st = 1\ . \qquad (2.7)$$

Then we have

$$\Delta^{M\times N/\Gamma}[(\Gamma(f) \circ p) \times (\Gamma(\varphi) \circ \pi)] = (\lambda+\mu)[(\Gamma(f) \circ p) \times (\Gamma(\varphi) \circ \pi)]\ . \qquad (2.8)$$

The relation (2.8) by means of (2.7) becomes

$$\Delta^{M\times N/\Gamma}[(tf \circ p) \times (s\varphi \circ \pi)] = (\lambda + \mu)[(tf \circ p) \times (s\varphi \circ \pi)]$$

or

$$\Delta^{M\times N/\Gamma}[(f\circ p)\times(\varphi\circ\pi)] = (\lambda+\mu)[(f\circ p)\times(\varphi\circ\pi)]\ . \quad (2.9)$$

From the relation (2.9) we obtain that $[(f\circ p)\times(\varphi\circ\pi)]$ is an eigenfunction of $\Delta^{M\times N/\Gamma}$ for the Riemannian manifold $(M\times N/\Gamma,\ g\times h/\Gamma)$ with eigenvalue $\lambda+\mu$.

We denote by $H^t(M)$ and $H^s(N)$ the subspaces of $Q_\lambda(M,g)$ and $Q_\mu(N,h)$ which are defined by

$$H^t_\lambda(M) = \{f\in Q_\lambda(M,g)/\Gamma(f) = tf,\ t\in C\}$$

$$H^s_\mu(N) = \{\varphi\in Q_\mu(N,h)/\Gamma(\varphi) = s\varphi,\ s\in C\} \quad (2.10)$$

respectively. The multiplicity of the eigenvalue of $k = \lambda+\mu$ for the $\Delta^{M\times N/\Gamma}$ is

$$m(k) = \sum_{\substack{\lambda+\mu+k \\ s,t\in C,\, st=1}} \dim H^t_\lambda(M)\cdot\dim H^s_\mu(N)\ . \quad (2.11)$$

If $m(\lambda+\mu) = 0$, then $\lambda+\mu$ is not eigenvalue for the Laplace operator $\Delta^{M\times N/\Gamma}$.

3. We consider the Euclidean space $\mathbf{R}^{n+1}$ with the standard coordinate system $(x_1,\dots,x_{n+1})$. We denote by $\Delta^{\mathbf{R}^{n+1}}$ the Laplace operator on $\mathbf{R}^{n+1}$ which has the form

$$\Delta^{\mathbf{R}^{n+1}} = -\left(\frac{\partial^2}{\partial x_1^2}+\dots+\frac{\partial^2}{\partial x_{n+1}^2}\right)\ .$$

We denote by P^k the space of complex homogeneous polynomials of degree k on $\mathbf{R}^{n+1}$. Let H^k be subspace of P^k which defined by

$$H^k = \{f\in P^k/\Delta^{\mathbf{R}^{n+1}}f = 0\}$$

consisting of harmonic polynomials on $\mathbf{R}^{n+1}$. We denote by

$$r^2 = \sum_{l=1}^{n+1} x_1^2\ .$$

It can be easily seen that the group $O(n+1)$ acts on P^k. The space H^k is a Γ-invariant subspace of P^k and r^2 is a Γ-invariant element of P^2.

The following decomposition is valid ([1], p. 161)

$$P^k = H^k + r^2 P^{k-2} , \tag{3.1}$$

where we put $P^{-1} = P^{-2} = \{0\}$.

We consider the inclusion map

$$i : S^n \to \mathbf{R}^{n+1} , \; i : P \to i(P) = P , \forall P \in S^n$$

which induces a map

$$i^* : C^\infty(\mathbf{R}^{n+1}) \to C^\infty(S^n), \; i^* : f \to i^*(f) = f \circ i .$$

The orthogonal group $O(n+1)$ acts naturally on the spaces $C^\infty(\mathbf{R}^{n+1})$ and $C^\infty(S^n)$ and its actions commute with the map i^*.

It is known that the restriction of an element f of H^k on S^n gives one of the k-eigenfunctions on S^n. Therefore the eigenspace Λ^{λ_k}, which will be denoted from now on by Λ^k, is defined by

$$\Lambda^k = \{ f/_{S^n} / f \in H^k \}$$

where the k-eigenvalue of Δ^{S^n} is $\lambda_k = k(k+n-1)$.

It can be easly seen that the map i^* gives an $O(n+1)$-isomorphism of H^k onto Λ^k.

From the spaces Λ^k and H^k we construct the following subspaces

$$\Lambda^{k,s} = \{ f \in \Lambda^k / \Gamma(f) = sf \}, s \in C, |s| = 1 \tag{3.2}$$

$$H^{k,s} = \{ f_1 \in H^k / \Gamma(f_1) = sf_1 \}, s \in C, |s| = 1 . \tag{3.3}$$

Since i^* is an $O(n+1)$-isomorphism between $\Lambda^{k,s}$ and $H^{k,s}$ and Γ is a subgroup of $O(n+1)$ we conclude that the $H^{k,s}$ and $\Lambda^{k,s}$ are isomorphic.

Therefore the study of the space $\Lambda^{k,s}$ is reduced to the study of the space $H^{k,s}$.

For $s = 1$ we obtain the spaces $H^{k,1}$ and $\Lambda^{k,1}$, which are denoted by H^k_Γ and Λ^k_Γ consisting of all Γ-invariant elements of H^k and Λ^k respectively.

We consider the natural projection

$$\pi : S^n \to S^n/\Gamma$$

which induces the injective map

$$\pi^* : C^\infty(S^n/\Gamma) \to C^\infty(S^n),\ \pi^*(f) = f \circ \pi$$

and the subspace $\pi^*(C^\infty(S^n/\Gamma))$ of $C^\infty(S^n)$ contains all the functions on S^n which are invariant by the action of the group Γ. It can be easily obtained

$$\Delta^{S^n}\pi^*(f) = \pi^*(\Delta^{S^n}/f),\ \forall \in C^\infty(S^n/\Gamma)$$

which yields that the space $(\pi^*)^{-1}\Lambda_\Gamma^k$ is the eigenspace of $\Delta^{S^n/\Gamma}$ with eigenvalue $k(k+n-1)$ and of course isomorphic to H_Γ^k.

Now, in order to compute $Sp(S^n \times S^m/\Gamma,\ g_0 \times g_0'/\Gamma)$, which will be denoted $Sp(S^n \times S^m/\Gamma)$, we distinguish two cases.

We assume that the group Γ has two elements that means $\Gamma = \mathbf{Z}_2$. The action of Γ on the vector space H^k gives

$$\Gamma(f) = f, \text{ if } f \in H^k \text{ and } k \text{ is even}$$

$$\Gamma(f) = -f, \text{ if } f \in H^k \text{ and } k \text{ is odd}$$

and therefore

$\Gamma(\varphi) = \varphi$, if $\varphi \in \Lambda^k$ and k is even and $\Gamma(\varphi) = -\varphi$, if $\varphi \in \Lambda^k$. From the above we conclude that

$$\Lambda^{k,1} = \Lambda^k, \text{ if } k \text{ is even, and } \Lambda^{k,1} = \{0\}, \text{ if } k \text{ is odd} \tag{3.4}$$

$$\Lambda^{k,-1} = \{0\}, \text{ if } k \text{ is even, and } \Lambda^{k,-1} = \Lambda^k, \text{ if } k \text{ is odd}\ . \tag{3.5}$$

The relations (3.4) and (3.5) and the theorem 2.1 yield the following: If $f \in \Lambda^k(S^n)$, $\varphi \in \Lambda^v(S^m)$ and k, v are simultaneously even numbers, then $(f \circ \rho) \times (\varphi \circ \pi)$ is an eigenfunction of $\Delta^{S^n \times S^m/\Gamma}$ with eigenvalue $p = k(k+n-1) + v(v+m-1)$ and its multiplicity is given by

$$\sum_{p=k(k+n-1)+v(v+m-1)} \dim \Lambda^k (S^n) \cdot \dim \Lambda^v(S^m)$$

$$= \Sigma \frac{(n+k-2)\dots(n+1)n(n+2k-1)}{k!} \cdot$$

$$\cdot \frac{(m+v-2)\dots(m+1)m(m+2v-1)}{v!}$$

where p and π the standard projections

$$\begin{array}{ccc} & S^n \times S^m & \\ p \swarrow & & \searrow \pi \\ S^n & & S^m \end{array}$$

This is obtained from the fact that $\Gamma(f) = f$, with $t = 1$ and $\Gamma(\varphi) = \varphi$ with $s = 1$, which give $st = 1$.

The same is true when k and v are both odd, then we have $\Gamma(f) = -f$ with $t = -1$ and $\Gamma(\varphi)$ with $s = -1$ which imply $st = 1$.

We have proved the theorem:

THEOREM 3.1. *The $Sp(S^n \times S^m/\mathbf{Z}_2)$ has the following form $p = k(k+n-1) + v(v+m-1)$ with multiplicity,* $\sum \frac{(n+k-2)\dots(n+1)n}{k!}(n+2k-1) \cdot \frac{(m+v-2)\dots(m+1)m}{v!}(m+2v-1)$ $(p = k(k+n-1) + v(v+m-1)$ *where k, v are both even or odd, $k = 0, 1, \dots, v = 0, 1, \dots$.*

The other case is when the group Γ has order greater or equal than 3, that means $\Gamma \simeq \mathbf{Z}_q$, $q \geq 3$. In this case in order to have meaning the problem the dimension of the spheres must be odd. Therefore we have to compute the $Sp(S^{2n-1} \times S^{2m-1}/\Gamma)$.

Let γ be a generator of the group Γ which is an element of $SO(2n) \times SO(2m)$ and therefore has eigenvalue of the form

$$\delta^{p_1}, \dots, \delta^{p_n}, \delta^{p_{n+1}}, \dots, \delta^{p_{n+m}}, \delta^{-p_1}, \dots, \delta^{-p_n}, \delta^{-p_{n+1}}, \dots, \delta^{p_{n+m}}$$

where $\delta = \exp \frac{2\pi\sqrt{-1}}{q}$. This group has the form

$$\Gamma = \{\gamma^l\}_{l=0}^{q-1} .$$

Let $(x_1, y_1, \dots, x_n, y_n)$ and $(u_1, v_1, \dots, u_m, v_m)$ be the standard coordinate systems on $\mathbf{R}^{2n}$ and $\mathbf{R}^{2m}$ respectively. We introduce the complex coordinate systems

$$z_i = x_i + \sqrt{-1}y_i, i = 1, \dots, n \text{ and } w_j = u_j + \sqrt{-1}v_j \ , \ j = 1, \dots, m$$

from which we obtain

$$x_i = \frac{1}{2}(z_i + \bar{z}_i),\ y_i = \frac{1}{2\sqrt{-1}}(z_i - \bar{z}_i),$$

$$u_j = \frac{1}{2}(w_j + \bar{w}_j),\ v_j = \frac{1}{2\sqrt{-1}}(w_j - \bar{w}_j)$$

and therefore $(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n, w_1, \bar{w}_1, \ldots, w_m, \bar{w}_m)$ can be taken as a coordinate system on $\mathbf{R}^{2n} \times \mathbf{R}^{2m} = \mathbf{R}^{2(n+m)}$.

Hence all the monomials of the form

$$Z^I \bar{Z}^J W^S \bar{W}^T = (z_1)^{i_1} \ldots (z_n)^{i_n} (\bar{z}_1)^{j_1} \ldots (\bar{z}_n)^{j_n}$$

$$(w_1)^{s_1} \ldots (w_m)^{s_m} (\bar{w}_1)^{t_1} \ldots (\bar{w}_m)^{t_m},$$

where $i_1, \ldots, i_n, j_1, \ldots, j_n,\ s_1, \ldots, s_m,\ t_1, \ldots, t_m \geq 0$ and $i_1 + \ldots + i_n + j_1 + \ldots + j_n + s_1 + \ldots + s_m + t_1 + \ldots + t_m = k$, form a base for all the homogeneous polynomials P^k of degree k on $\mathbf{R}^{2(n+m)}$.

The action of the element γ^l of the group Γ on the monomial $Z^I \bar{Z}^J W^S \bar{W}^T$ gives

$$\gamma^l(Z^I \bar{Z}^J W^S \bar{W}^T) = \tag{3.6}$$
$$= \delta^{l(i_1 p_1 + \ldots + i_n p_n - j_1 p_1 \ldots - j_n p_n + s_1 p_{n+1} + \ldots + s_m p_{n+m} - t_1 p_{n+1} - \ldots - t_m p_{n+m})} (Z^I \bar{Z}^J W^S \bar{W}^T).$$

We denote by

$$r = l(i_1 p_1 + \ldots + i_n p_n - j_1 p_1 - \ldots - j_n p_n + s_1 p_{n+1} + \ldots + s_m p_{n+m}$$
$$- t_1 p_{n+1} - \ldots - t_m p_{n+m})$$

and therefore we have

$$r = \sigma(\mathrm{mod}\, q) \text{ and } \sigma \text{ one of the numbers } 0, 1, \ldots, q-1. \tag{3.7}$$

The relation (3.6) by means of (3.7) takes the form

$$\gamma^l(Z^I \bar{Z}^J W^S \bar{W}^T) = \delta^\sigma (Z^I \bar{Z}^J W^S \bar{W}^T),\ \sigma \text{ one } of\ 0, 1, \ldots,\ q-1. \tag{3.8}$$

Similarly we have

$$\gamma^l(Z^I\bar{Z}^J) = \delta^\kappa(Z^I\bar{Z}^J),\ \gamma^l(W^S\bar{W}^T) = \delta^\tau(W^S\bar{W}^T) \qquad (3.9)$$

for appropriate $\kappa, \tau \in 0, \dots, q-1$.

The action of the group Γ on the vector spaces $H^k(\mathbf{R}^{2n})$ and $H^v(\mathbf{R}^{2m})$ gives the following subspaces

$$H^{k,\sigma} = \{f \in H^k(\mathbf{R}^{2n})/\Gamma(f) = \delta^\sigma f\}\ ,\ \sigma = 0, 1, \dots, q-1 \quad (3.10)$$

$$H^{v,\tau} = \{\varphi \in H^v(\mathbf{R}^{2m})/\Gamma(\varphi) = \delta^\tau \varphi\}\ ,\ \tau = 0, 1, \dots, q-1\ . \quad (3.11)$$

Therefore the vector spaces $H^k(\mathbf{R}^{2n})$ and $H^v(\mathbf{R}^{2m})$ can be decomposed as follows

$$H^k(\mathbf{R}^{2n}) = H^{k,0} \oplus H^{k,1} \oplus \dots \oplus H^{k,q-1} \qquad (3.12)$$

$$H^v(\mathbf{R}^{2m}) = H^{v,0} \oplus H^{v,1} \oplus \dots \oplus H^{v,q-1}\ . \qquad (3.13)$$

The same is true for the vector spaces $\Lambda^k(S^{2n-1})$ and $\Lambda^v(S^{2m-1})$, that is we have

$$\Lambda^k(S^{2n-1}) = \Lambda^{k,0} \oplus \Lambda^{k,1} \oplus \dots \oplus \Lambda^{k,q-1} \qquad (3.14)$$

$$\Lambda^v(S^{2m-1}) = \Lambda^{v,0} \oplus \Lambda^{v,1} \oplus \dots \oplus \Lambda^{v,q-1} \qquad (3.15)$$

where

$$\Lambda^{k,\sigma}(S^{2n-1}) = \{f \in \Lambda^k(S^{2n-1})/\Gamma(f) = \delta^\sigma f\},\ \sigma = 0, 1, \dots, q-1 \qquad (3.16)$$

$$\Lambda^{v,\tau}(S^{2m-1}) = \{\varphi \in \Lambda^v(S^{2m-1})/\Gamma(\varphi) = \delta^\tau \varphi\},\ \tau = 0, 1, \dots, q-1 \qquad (3.17)$$

From the construction of these spaces we obtain

$$H^{k,\sigma}(\mathbf{R}^{2n}) = \Lambda^{k,\sigma}(S^{2n-1}),\ \sigma = 0, \dots, q-1,$$
$$H^{v,\tau}(\mathbf{R}^{2m}) \simeq \Lambda^{v,\tau}(S^{2m-1}),\ \tau = 0, \dots, q-1\ . \qquad (3.18)$$

If $f \in \Lambda^{k,\sigma}(S^{2n-1})$, $\varphi \in \Lambda^{v,\tau}(S^{2m-1})$ and $\sigma + \tau = 0 \operatorname{mod}(q)$ and $\dim(\Lambda^{k,\sigma}(S^{2n-1})) \neq 0$, $\dim(\Lambda^{v,\tau}(S^{2m-1})) \neq 0$, then $(f \circ p) \times (\rho \circ \pi)$ is an eigenfunction of $\Delta^{S^{2n-1} \times S^{2m-1}}/\Gamma$ with eigenvalue $\lambda(k, v) = k(k +$

$2n-2) + v(v+2m-2)$. Generally there are more than one pairs of integers $(k,v), (k',v'), \ldots$, with the property $\lambda(k,v) = \lambda(k',v') = \ldots$. Therefore the multiplicity of (k,v) is $m_{k,v} = \bar{m}_{k,v} + \bar{m}_{k',v'} + \ldots$. These $m_{k,v}, \ldots$ are computed as follows

$$\bar{m}_{k,v} = \dim \left(\bigoplus_{\substack{\sigma,\tau=0 \\ \sigma+\tau\equiv 0 \bmod (q)}}^{q-1} \Lambda^{k,\sigma}(S^{2n-1}) \otimes \Lambda^{v,\tau}(S^{2m-1}) \right) \tag{3.19}$$

which is equal to

$$\bar{m}_{k,v} = \dim \left(\bigoplus_{\substack{\sigma,\tau=0 \\ 0+\tau\equiv 0 \bmod (q)}}^{q-1} H^{k,\sigma}(S^{2n-1}) \otimes H^{v,\tau}(S^{2m-1}) \right)$$

$$= \sum_{\substack{\sigma,\tau=0 \\ \sigma+\tau\equiv \bmod (q)}}^{q-1} \dim H^{k,\sigma}(S^{2n-1}) \dim H^{v,\tau}(S^{2m-1}) . \tag{3.20}$$

From the above we conclude that to find, if $\lambda(k,v) = \lambda(k',v') = \ldots$ is an eigenvalue or not and its multiplicity it is enough to determine

$$\dim(H^{k,\sigma}(\mathbf{R}^{2n})),\ \dim(H^{v,\tau}(\mathbf{R}^{2m})),$$

$$\dim(H^{k',\sigma}(\mathbf{R}^{2n})),\ \dim(H^{v'\sigma}(\mathbf{R}^{2m})) .$$

We give the method to compute $\dim(H^{k,\sigma}(\mathbf{R}^{2n}))$ similarly we estimate $\dim(H^{v',\tau}(\mathbf{R}^{2m}))$ and the others.

We denote by x_k and $\tilde{x}_k$ the characters of the natural representations of $SO(2n)$ on H^k and P^k respectively. Then it can be easily obtained

$$\dim H^{k,\sigma} = \frac{1}{|\Gamma|} \sum_{l=0}^{q-1} \sigma^1 x_k(\gamma^l),\ \sigma = 0,1,2,\ldots,q-1 . \tag{3.21}$$

Since r^2 is invariant under the action of the group $SO(2n)$, then from the relation (3.1) we take

$$x_k(\gamma^l) = \tilde{x}_k(\gamma^l) - \tilde{x}_{k-2}(\gamma^l) . \tag{3.22}$$

We construct the function $\psi(z)$ of the complex variable z defined by

$$\psi(z) = \sum_{k=0}^{\infty} x_k(\gamma^l) z^k \tag{3.23}$$

which by means of (3.22) takes the form

$$\psi(z) = \sum_{k=0}^{\infty} [\tilde{x}_k(\gamma^l) - \tilde{x}_{k-2}(\gamma^l)] z^k . \tag{3.24}$$

But we have $\sum_{k=0}^{\infty} \tilde{x}_{k-2}(\gamma^l) z^k = \sum_{k=0}^{\infty} \tilde{x}_k(\gamma^l) z^{k+2}$ and hence (3.24) implies

$$\psi(z) = (1 - z^2) \sum_{k=0}^{\infty} \tilde{x}_k(\gamma^l) z^k . \tag{3.25}$$

From the relation (3.6), if we obtain only Z and $\bar{Z}$, we have

$$\begin{gathered} \gamma^l(Z^I \bar{Z}^J) = \delta^{l(i_1 p_1 + \ldots + i_n p_n - j_1 p_1 - \ldots - j_n p_n)} (Z^I \bar{Z}^J) \\ i_1 + \ldots + i_n + j_1 + \ldots + j_n = k \end{gathered} \tag{3.26}$$

which implies

$$\tilde{x}_k(\gamma^l) = \sum_{i_1 + \ldots + i_n + j_1 + \ldots + j_n = k} \delta^{l(i_1 p_1 + \ldots + i_n p_n - j_1 p_1 - \ldots - j_n p_n)} . \tag{3.27}$$

The function (3.25) by means (3.27) implies

$$\psi(z) = (1 - z^2) \sum_{k=0}^{\infty} \sum_{i_1 + \ldots + i_n + j_1 + \ldots + j_n = k} \delta^{l(i_1 p_1 + \ldots + i_n p_n - j_1 p_1 - \ldots - j_n p_n)} z^k . \tag{3.28}$$

We consider the function

$$\Phi(z) = \frac{(1 - z^2)}{\prod_{i=1}^{n} (1 - \delta^{l p_i} z)(1 - \delta^{-l p_i} z)} \tag{3.29}$$

whose expansion on $D^1 = \{z \in C/|z| < 1\}$ gives (3.28) and hence

$$\psi(z) = \Phi(z) = \frac{(1-z^2)}{\prod_{i=1}^{n}(1-\delta^{lp_i}z)(1-\delta^{-lp_i}z)} . \tag{3.30}$$

We construct the following function

$$F_\sigma(z) = \sum_{k=0}^{\infty} \dim H^{k,\sigma} z^k \tag{3.31}$$

which by means of (3.21), (3.23) and (3.30) takes the form

$$F_\sigma(z) = \frac{1}{|\Gamma|}\sum_{l=0}^{q-1} \frac{\delta^\sigma(1-z^2)}{\prod_{i=1}^{n}(1-\delta^{lp_i}z)(1-\delta^{-lp_i}z)} . \tag{3.32}$$

Therefore the coefficient of z^k in the expansion of the function (3.32) on $D^1 = \{z \in C/|z| < 1\}$ gives the dimension of $H^{k,\sigma}$.

Now, we consider the function

$$F(z) = \sum_{\substack{\sigma,\tau \\ \sigma+\tau = 0 \bmod (q)}}^{0,q-1} F_\sigma(z) \cdot F_\tau(z) \tag{3.33}$$

which by means of (3.32) becomes

$$F(z) = \frac{1}{|\Gamma|}\sum_{l=0}^{q-1} \frac{(1-z^2)^2}{\prod_{i=1}^{n}(1-\gamma^{lp_i}z)(1-\gamma^{-lp_i}z) \cdot \prod_{j=1}^{m}(1-\gamma^{lp_{n+j}}z)(1-\gamma^{-lp_{n+j}})} . \tag{3.34}$$

From the above we conclude that the eigenvalue $\lambda(k,v) = \lambda(k',v') = k(k+2n-2)+v(v+2m-2)$ exists for $\Delta^{S^{2n-1}\times S^{2m-1}/\Gamma}$ if the coefficient $z^{k+v}, z^{k'+v'}, \ldots$, in the expansion of (3.34) on $D^1 = \{z \in C/|z| < 1\}$ are different from zero, otherwise there exists no such eigenvalue. The sum of these coefficients is the multiplicity of the eigenvalue $k(k+2n-2) + v(v+2m-2)$.

Now, we can state the following theorem.

THEOREM 3.3. *Let* Γ *be a finite cyclic group of order* $q \geq 3$ *which acts on* $S^{2n-1} \times S^{2m-1}$ *fixedpointfreely. The eigenvalues of* $\Delta^{S^{2n-1} \times S^{2m-1}/\Gamma}$ *has the form* $(k,v) = (k',v') = \ldots = k(k+2n-2)+v(v+2m-2)$, *where* $k = 0,1,\ldots, v = 0,1,\ldots$ *with multiplicity* $m_{k,v} = \bar{m}_{k,v} + \bar{m}_{k',v'} + \ldots$, *where* $\bar{m}_{k,v}, \bar{m}_{k'v'}, \ldots$, *are the coefficients of* $z^{k+v}, z^{k'+v'}, \ldots$ *in the expansion on* $D^1 = \{z \in C / |z| < 1\}$ *of (3.24).*

4. We consider in this paragraph that Γ is a cyclic group of order 3. The function (3.34) after some estimates becomes

$$F(z) = \frac{(1-z^2)^2}{3}\left[\frac{1}{(1-z)^{2(n+m)}} + \frac{2(1-z)^{n+m}}{(1-z^3)^{n+m}}\right] . \tag{4.1}$$

It is known

$$\frac{1}{(1-z)^{2(n+m)}} = \sum_{k=0}^{\infty}\binom{2(n+m)-1+k}{2(n+m)-1} z^k \tag{4.2}$$

$$\frac{1}{(1-z^3)^{n+m}} = \sum_{k=0}^{\infty}\binom{n+m-1+k}{n+m-1} z^{3k} \tag{4.3}$$

$$(1-z)^{n+m} = \sum_{k=0}^{n+m}(-1)^k\binom{n+m}{k} z^k . \tag{4.4}$$

The function (4.1) by means of (4.2), (4.3) and (4.4) takes the form

$$F(z) = \frac{(1-z^2)^2}{3}\left[\sum_{k=0}^{\infty}\left[\binom{2(n+m)-1+k}{2(n+m)-1} z^k + \right.\right.$$

$$\left.+2\left(\sum_{k=0}^{n+m}(-1)^k\binom{n+m}{k} z^k\right)\left(\sum_{k=0}^{\infty}\binom{n+m-1+k}{n+m-1} z^{3k}\right)\right] .$$

REFERENCES

[1] BERGER M., GAUDACHON P. and MAZET E., *Le spectre d'une variété Riemannienne*, Lecture Notes in Mathematics *194*, Springer-Verlag, Berlin-Heidelberg-New York (1971).

[2] CURTIS C. and REINER I., *Representation theory of finite groups and associative algebras*, Pure and Applied Mathematics, Vol. XI, Interscience Publishers (1962).

[3] IKEDA A., *On the spectrum of a compact riemannian manifold of positive constant curvature*, Osaka J.Mat.*19* (1980) 75-93.