ON TRANSFINITE SPLITTINGS OF INFINITE SETS (*)

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SOMMARIO. - *Dimostriamo qui un teorema fondamentale concernente la divisione iterata di un sottoinsieme proprio e non vuoto di un dato insieme (in particolare, un numero cardinale infinito) in non più di due sottoinsiemi non vuoti e disgiunti. Da questo teorema si può enunciare l'esistenza di alberi con una grande altezza cardinale con proprietà non banali.*

SUMMARY. - *We prove below a rather basic Theorem concerning the iterative splitting of a nonempty proper subset of a set (specifically, an infinite cardinal number) into at most two nonempty disjoint subsets. Based on this, the existence of trees with large cardinal heights having some nontrivial properties can be established.*

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Let $K = \{0,1,2,\ldots,\omega,\ \omega+1,\ldots,\omega 2,\ldots,\omega^2,\ldots,\omega_1,\ldots\}$ be an infinite cardinal. We consider [1, p. 68] a tree $(T,\supseteq)$ whose elements are nonempty subsets of K.

The tree $(T,\supseteq)$ is formed as follows:

The 0-th level L_0 of T consists of the single element K.

The 1-st level L_1 of T consists either of the single element

$$K - \{\min K\} = K - \{0\} = \{1,2,\ldots,\omega,\ldots,\omega_1,\ldots\}$$

or else of two elements into which $K - \{\min K\}$ is split (in any manner whatsoever) into two nonempty disjoint subsets, say as

$$\{1,2,3,15,\ldots,\omega+1,\ldots\} \quad \text{and} \quad \{4,5,6,\ldots,\omega^2,\ldots\}$$

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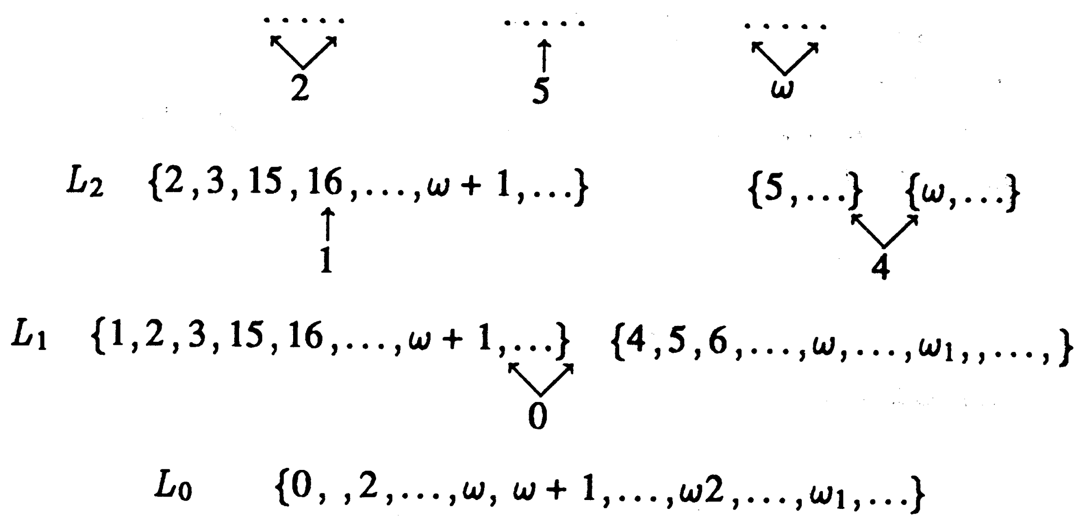
Thus, the element on level L_0 has at most two immediate successors (w.r.t. $\supseteq$) on level L_1.

The 2-nd level L_2 of T consists of at most four elements which are at most four nonempty disjoint subsets of K and which are the immediate successors of the elements of L_1 in the manner in which the elements of L_1 are the immediate successors of the element of L_0.

The above can be shown diagramatically as:



As expected, for every natural number $n \in \omega$, the level L_{n+1} is formed from the level L_n the way that L_1 is formed from L_0 or L_2 is formed from L_1.

The first infinite level L_ω of T consists of elements which are nonempty intersections of the form $\bigcap_{i \in \omega} S_i$ with $S_i \in L_i$.

In general (as mentioned in the Definition below) for every ordinal number u, the level L_u of T is formed motivated by the formation of L_0, L_1, L_2 and L_ω.

We observe that it does not matter that in the above the manner in which an element of L_u is split is not specified. Without loss of generality we could have considered an initially given fixed splitting of all the subsets of K and then refer to that splitting in the formation of L_{u+1}. However, we have not done so in order not to complicate the notation.

Let us observe that it is more or less obvious that the above splitting process must terminate at some level. Consequently, we have $L_t = \emptyset$ for some $t \leq K$. However, it is not obvious that if $L_t = \emptyset$ for some $t \leq K$, then every element of K must have been used as a minimum of some element

S_i of the level L_i with $i < t$. In fact, the Theorem below shows that this is the case.

In what follows, if S is a nonempty set of ordinals then min S (as already used in the above) stands for the *minimum* (i.e., the smallest) element of S.

DEFINITION. *Let $K > 0$ be a (finite or infinite) cardinal number. We consider a tree $(T, \supseteq)$ whose elements are nonempty subsets of K where for every ordinal u the level L_u of T is formed as follows:*

(1) $L_0 = \{K\}$

(2) if $S \in L_u$, then S has at most two immediate successors S' and S'' in L_{u+1} such that

$$S' \cup S'' = S - \{\min S\} \quad \textit{and} \quad S' \cap S'' = \emptyset$$

$$\textit{and } S' \neq \emptyset \textit{ and } S'' \neq \emptyset \ .$$

(3) if u is a limit ordinal then

$$L_u = \left\{ \bigcap_{i<u} S_i | S_i \in L_i \textit{ and } \bigcap_{i<u} S_i \neq \emptyset \right\}$$

It is easy to verify that $(T, \supseteq)$ is a tree and that

(4) *the height of T is $\leq K$*

Indeed, from (1) to (3) it follows that if $S \in L_u$ then the set of all the predecessors of S in $(T, \supseteq)$ form a well ordered set of type u. Thus, $(T, \supseteq)$ is a tree. On the other hand, (4) follows from the fact that the tallest possible tree is the well ordered (by $\supseteq$) tree

$$K, K-1,\ K-2, \ldots, K-u, \ldots \quad \textit{for every } u < K \ .$$

The above example shows that for every (finite or infinite) cardinal $K > 0$ there exists a tree satisfying the above Definition. However, it can be also shown that there are trees $(T, \supseteq)$ whose heights are strictly less than K.

THEOREM. *Let K, T and L_u be as in the definition above. Then*

$$\overline{\overline{L_u}} \leq 2^{\overline{\overline{u}}} \textit{ for every } u < K \ . \tag{5}$$

Moreover,

(6) if $L_t = \emptyset$ for some $t < K$ then for every $v < K$ it is the case that $v = \min S$ for some $S \in T$

and

(7) $$\overline{\overline{\bigcup_{u<t} L_u}} = K$$

Proof. From (1) to (3) it follows that every element at the level L_u can be identified with a dyadic sequence (i.e., a sequence made of 0's or 1's) of the ordinal type u. Thus $\overline{\overline{L_u}} \leq 2^{\overline{\overline{u}}}$ and (5) is established.

To prove (6), let $L_t = \emptyset$ for some $t \leq K$.

Let us assume to the contrary that there exists $v < K$ such that v is not the minimum of any of the elements of the tree T.

Under the above assumption, we first prove that

(8) $$v \in \cup L_u \quad for\ every \quad u < t \quad for\ which\ L_u \neq \emptyset$$

i.e., every nonempty level L_u has an element S_u (a subset of K) such that $v \in S_u$. If not, let $s < t$ be the smallest ordinal for which (8) fails, i.e.

(9) $$v \notin \cup L_s \quad with \quad L_s \neq \emptyset\,.$$

Clearly, $s \neq 0$ since $\cup L_0 = K$ by (1) which would imply $v \in \cup L_0$, contradicting (9).

Moreover, $s \neq h + 1$ (i.e. s cannot be a nonzero nonlimit ordinal). This is because otherwise $v \in L_h$ and since $v \notin L_{h+1}$ it would follow from (2) that v is the minimum of an element of L_h, contradicting the above assumption.

Furthermore, s cannot be a limit ordinal. This is because otherwise for every $u < s$ there would be an element S_u of L_u such that $v \in S_u$ and therefore $v \in \bigcap_{u<s} S_u$. But then, from (3), it would follow that $v \in \cup L_s$, contradicting (9).

Thus, (8) is established.

To complete the proof of (6), it suffices to show that, in view of (8), our assumption above leads to a contradiction.

Based on the hypothesis of (6), let $s \leq K$ be the smallest ordinal number for which $L_s = \emptyset$. Our assumption above is that there exists $v < K$ such that v is not the minimum of any of the elements of T. Under this assumption, from (8) it followed that $v \in \cup L_u$ for every $u < s$. Clearly, $s \neq 0$ since $L_0 = \{K\} \neq \emptyset$. Again, $s \neq h+1$ since otherwise (by reasoning as in the above) v would be the minimum of an element of L_h. Again, s cannot be a limit ordinal since otherwise (by reasoning as in the above) we would have $v \in \cup L_s$ contradicting that $\cup L_s = \emptyset$.

Thus, our assumption above is false and (6) is established.

Finally, to prove (7), we observe that distinct element of T are nonempty subsets of K with distinct minimums. By (6) every element of K is the minimum of some element of T. Thus, there exists a one-to-one correspondence between K and $\bigcup_{u<1} L_u$ which implies (7).

Thus, the Theorem is proved.

REMARK 1. Let us observe that if K is a *strong limit* cardinal (i.e., $t < K$ implies $2^{\overline{\overline{t}}} < K$) then from (5) and (7) it follows that in (6) we must have $t = K$. This is because otherwise we would have $K = \sum_{u<t} 2^{\overline{\overline{u}}} \leq \overline{\overline{t}}\, 2^{\overline{\overline{t}}} < K$, which is a contradiction.

REMARK 2. From Remark 1 it follows that for a large (such as strong limit, strongly inaccessible, etc.) cardinal K the Theorem above ensures the existence of a tree (w.r.t. $\supseteq$) of height K whose elements are rather arbitrary nonempty subsets of K and whose levels L_u may even have $2^{\overline{\overline{u}}}$ elements. This kind of trees play a significant role in the partition relations for cardinals (cf. [2],[3]).

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