ON SOME TOEPLITZ AND HAENKEL MATRICES WITH ELEMENTS 0,1 AND -1 (*)

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SOMMARIO. - *Scopo di questo lavoro è dare un contributo alla raccolta di matrici test. Per questo motivo vengono considerate due classi di matrici, di Haenkel e di Toeplitz, delle quali si calcola l'inversa. Inoltre, se n è l'ordine della matrice, vengono calcolati $[n/2]$ autovalori, e i corrispondenti autovettori, esattamente.*

SUMMARY. - *The aim of this paper is to give a contribution to the collection of test matrices. To this end, two classes of Toeplitz and Haenkel matrices are taken into account, of which inverse is calculated. Moreover, if n is the order, $[n/2]$ eigenvalues and the corresponding eigenvectors are given exactly.*

1. Introduction.

In this paper we study the properties of two classes of Toeplitz matrices, with elements 0, 1 and -1, of the two types

$$A = \begin{bmatrix} 1 & 0 & 1 & \cdot & \cdot & 1 & 1 & 1 \\ 0 & 1 & 0 & \cdot & \cdot & 1 & 1 & 1 \\ 1 & 0 & 1 & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & 1 & 0 & 1 \\ 1 & 1 & 1 & \cdot & \cdot & 0 & 1 & 0 \\ 1 & 1 & 1 & \cdot & \cdot & 1 & 0 & 1 \end{bmatrix} .$$

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and

$$B = \begin{bmatrix} 1 & 0 & -1 & \cdot & \cdot & -1 & -1 & -1 \\ 0 & 1 & 0 & \cdot & \cdot & -1 & -1 & -1 \\ -1 & 0 & 1 & \cdot & \cdot & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & 1 & 0 & -1 \\ -1 & -1 & -1 & \cdot & \cdot & 0 & 1 & 0 \\ -1 & -1 & -1 & \cdot & \cdot & -1 & 0 & 1 \end{bmatrix}.$$

If n is the order of the matrix, then $[n/2]$ eigenvalues are exactly calculated, while for each of the others an interval is given, including it.

Furthermore it is shown that the inverse of B has integer elements and the inverse of A is block-structured, with integers elements too (both apart a multiplicative coefficient).

In the last part it is shown how from the knowledge of some properties of Toeplitz matrices it is possible to carry out analogous results for two classes of Haenkel matrices.

2. Notations.

In the following we assume

$$N = \begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 & 1 \\ 1 & 1 & 1 & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & 1 & 1 \\ 0 & 1 & 1 & \cdot & \cdot & 1 & 1 \end{bmatrix};$$

$$T = \begin{bmatrix} 0 & -1 & 0 & \cdot & \cdot & 0 & 0 & 0 \\ -1 & 0 & -1 & \cdot & \cdot & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 & -1 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & -1 & 0 & -1 \\ 0 & 0 & 0 & \cdot & \cdot & 0 & -1 & 0 \end{bmatrix};$$

$J = [e_n, e_{n-1}, \ldots, e_2, e_1]$ (1) ; $u = [1,1,\ldots,1]^T$; $U = u\,u^T$;

$v = [1,1,\ldots,1,0]^T$; $z = [1,0,\ldots,0,0]^T$; $w = [0,0,\ldots,0,1]^T$;

(the dimension will be the suitable one).

3. Eigenvalues of matrix A.

We consider separately the two cases of n even or odd.

$n = 2k$

Matrix A can be written

$$A = \begin{bmatrix} M & N \\ JNJ & M \end{bmatrix}$$

where $M = A$ (of order k).

Transforming A with the use of matrix $S_1 = \begin{bmatrix} I & J \\ J & -I \end{bmatrix}$, whose inverse is $S_1^{-1} = \frac{1}{2} S_1$, we obtain

$$S_1^{-1} A S_1 = \begin{bmatrix} M + NJ & 0 \\ 0 & M - JN \end{bmatrix};$$

so eigenvalues of matrix A are the same of eigenvalues of the two diagonal blocks $M + NJ$ and $M - JN$.

The block $M - JN = T + zz^T$ is a tridiagonal matrix and so its eigenvalues and eigenvectors are (see [4])

$$\lambda_s = -2\cos\frac{2s}{2k+1}\pi, \quad s = 1,2,\ldots,k,$$

$$x_j^{(s)} = \frac{2}{\sqrt{2k+1}} \sin\frac{(2j-1)s}{2k+1}\pi, \quad j = 1,2,\ldots,k; \quad s = 1,2,\ldots,k.$$

The block $M + NJ$ is given by $M + NJ = T - ww^T + 2U$. The eigenvalues μ_s, $s = 1,2,\ldots,k$, of $T - ww^T$ are known (see [4]) therefore

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(1) e_i is the i-th element of canonical base in $\mathbf{R}^n$.

if $\mu_1 \le \mu_2 \le \ldots \le \mu_k$ the other eigenvalues of A are included into the intervals (see [3])

$$[\mu_r, \mu_{r+1}],\ r = 1,2,\ldots,k-1 \text{ and } [\mu_k, \mu_k + n]\ .$$

$n = 2k+1$

Matrix A can be written

$$A = \begin{bmatrix} M & v & U \\ v^T & 1 & v^T J \\ U & Jv & M \end{bmatrix}.$$

Transforming A with the use of matrix $S_2 = \begin{bmatrix} I & 0 & J \\ 0^T & \sqrt{2} & 0^T \\ J & 0 & -I \end{bmatrix}$ whose inverse is $S_2^{-1} = \frac{1}{2} S_2$, we obtain

$$S_2^{-1} A S_2 = \left[\begin{array}{cc|c} M+U & \sqrt{2}v & 0 \\ \sqrt{2}v^T & 1 & 0 \\ \hline 0 & 0 & M-U \end{array}\right];$$

so eigenvalues of matrix A are the same of eigenvalues of the two diagonal blocks of the transformed matrix.

The block $M-U$ is the tridiagonal matrix T, of order k, whose eigenvalues and eigenvectors are (see [4])

$$\lambda_s = -2\cos\frac{s}{k+1}\pi,\ s = 1,2,\ldots,k\,,$$

$$x_j^{(s)} = \sqrt{2/(k+1)}\sin\frac{js}{k+1}\pi,\ j = 1,2,\ldots,k;\ s = 1,2,\ldots,k\,.$$

REMARK 1. It is of interest to note that for k odd, i.e., for $n = 4r+3$ $(r = 0,1,\ldots)$, there is a zero eigenvalue for $s = (k+1)/2$; it follows that for such values the matrix A is singular.

The other diagonal block is the matrix of order $k+1$

$$\begin{bmatrix} 2 & 1 & 2 & \cdot & \cdot & 2 & 2 & \sqrt{2} \\ 1 & 2 & 1 & \cdot & \cdot & 2 & 2 & \sqrt{2} \\ 2 & 1 & 2 & \cdot & \cdot & 2 & 2 & \sqrt{2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & 2 & 2 & \cdot & \cdot & 2 & 1 & \sqrt{2} \\ 2 & 2 & 2 & \cdot & \cdot & 1 & 2 & 0 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & \cdot & \cdot & \sqrt{2} & 0 & 1 \end{bmatrix}$$

of which we know neither eigenvalues nor intervals of inclusion.

We can however write A in the form $A = T + U$. The eigenvalues μ_s, $s = 1, 2, \ldots n$, of T are known (see [4]) and therefore if $\mu_1 \leq \mu_2 \leq \ldots$ $\ldots \leq \mu_n$ the eigenvalues of A are included into the intervals (see [3])

$$[\mu_r, \mu_{r+1}], \; r = 1, 2, \ldots, n-1 \text{ and } [\mu_n, \mu_n + n] \; .$$

REMARK 2. Among these n intervals the k intervals including the $[n/2]$ eigenvalues, which are exactly known as eigenvalues of $M - U$, can be shut out.

Eigenvalues of matrix B.

We consider separately the two cases of n even or odd.

$\underline{n = 2k}$

Matrix B can be written

$$B = \begin{bmatrix} C & -N \\ -JNJ & C \end{bmatrix}$$

where $C = B$ (of order k).

Transforming B with the use of matrix S_1, we obtain

$$S_1^{-1} B S_1 = \begin{bmatrix} C - NJ & 0 \\ 0 & C + JN \end{bmatrix} \; ;$$

so eigenvalues of matrix B are the same of eigenvalues of the two diagonal blocks $C - NJ$ and $C + JN$.

The block $C + JN = 2I - T - zz^T$ is a tridiagonal matrix of order k and so its eigenvalues and eigenvectors are (see [4])

$$\lambda_s = 2 + 2\cos\frac{2s}{2k+1}\pi, \quad s = 1,2,\ldots,k\,,$$

$$x_j^{(s)} = \frac{2}{\sqrt{2k+1}}\sin\frac{(2j-1)s}{2k+1}\pi, \quad j = 1,2\ldots k;\ s = 1,2\ldots k\,.$$

The block $C - NJ$ is the matrix of order k

$$C - NJ = \begin{bmatrix} 0 & -1 & -2 & \cdot & \cdot & -2 & -2 & -2 \\ -1 & 0 & -1 & \cdot & \cdot & -2 & -2 & -2 \\ -2 & -1 & 0 & \cdot & \cdot & -2 & -2 & -2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -2 & -2 & -2 & \cdot & \cdot & 0 & -1 & -2 \\ -2 & -2 & -2 & \cdot & \cdot & -1 & 0 & -1 \\ -2 & -2 & -2 & \cdot & \cdot & -2 & -1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 1 & 0 & \cdot & \cdot & 0 & 0 & 0 \\ 1 & 2 & 1 & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 1 & 2 & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 2 & 1 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & 1 & 2 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & 0 & 1 & 2 \end{bmatrix} - 2U\,.$$

The eigenvalues μ_s, $s = 1,2,\ldots,k$, of the tridiagonal matrix are known (see [4]) and therefore if $\mu_1 \le \mu_2 \le \ldots \le \mu_k$ the other eigenvalues of B are included into the intervals (see [3])

$$[\mu_r, \mu_{r+1}],\ r = 1,2,\ldots,k-1 \text{ and } [\mu_k, \mu_k + n]\,.$$

$n = 2k + 1$

Matrix B can be written

$$B = \begin{bmatrix} C & -v & -U \\ -v^T & 1 & -v^T J \\ -U & -Jv & C \end{bmatrix}.$$

Transforming B with the use of matrix S_2 we obtain

$$S_2^{-1}BS_2 = \left[\begin{array}{cc|c} C-U & -\sqrt{2}v & 0 \\ -\sqrt{2}v^T & 1 & 0 \\ \hline 0 & 0 & C+U \end{array}\right] ;$$

so eigenvalues of matrix B are the same of eigenvalues of the two diagonal blocks of the transformed matrix.

The block $C + U = 2I - T$ is a tridiagonal matrix of order k whose eigenvalues and eigenvectors are (see [4])

$$\lambda_s = 2 + 2\cos\frac{s}{k+1}\pi, \quad s = 1,2,\ldots,k\,,$$

$$x_j^{(s)} = \sqrt{2/(k+1)}\sin\frac{js}{k+1}\pi, \quad j = 1,2\ldots k;\ s = 1,2\ldots k\,.$$

The other diagonal block is the matrix of order $k+1$

$$\begin{bmatrix} 0 & -1 & -2 & \cdot & \cdot & -2 & -2 & -\sqrt{2} \\ -1 & 0 & -1 & \cdot & \cdot & -2 & -2 & -\sqrt{2} \\ -2 & -1 & 0 & \cdot & \cdot & -2 & -2 & -\sqrt{2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -2 & -2 & -2 & \cdot & \cdot & 0 & -1 & -\sqrt{2} \\ -2 & -2 & -2 & \cdot & \cdot & -1 & 0 & 0 \\ -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & \cdot & \cdot & -\sqrt{2} & 0 & 1 \end{bmatrix}$$

of which we know neither eigenvalues nor intervals of inclusion. We can however write B in the form $B = 2I - T - U$. The eigenvalues μ_s, $s = 1,2,\ldots,n$, of $2I - T$ are known (see [4]) and therefore if $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$ the eigenvalues of B are included into the intervals (see [3])

$$[\mu_1 - n, \mu_1] \text{ and } [\mu_r, \mu_{r+1}],\ r = 1,2,\ldots,n-1\,.$$

REMARK 3. Among these n intervals the k intervals including the $[n/2]$ eigenvalues, which are exactly known as eigenvalues of $C + U$, can be shut out.

5. Matrix A^{-1}.

We have previously seen that $A = T + U = T + uu^T$: so if T is not singular and $1 + u^T T u \neq 0$, by the Sherman-Morrison formula we have

$$A^{-1} = T^{-1} - (T^{-1}uu^T T^{-1})/(1 + u^T T^{-1} u) .$$

Matrix T^{-1} exists only if its order is even (the eigenvalues of T are $\lambda_s = -2\cos(s\pi/(n+1)),\ s = 1,2\ldots n$).

From the knowledge of T^{-1} (see [1]) follows

$$1 + u^T T^{-1} u = \begin{cases} 1 - k & \text{if } k = 2r,\ r = 1,2,\ldots ,\\ -k & \text{if } k = 2r-1,\ r = 1,2,\ldots . \end{cases}$$

Hence A is not singular for $n = 2k$. By Remark 1, A is singular if $n = 4r+3,\ r = 0,1,\ldots$; for $n = 4r+1,\ r = 0,1,\ldots,$ A is invertible, as will be shown by the constructive proof of Theorem 3.

$\underline{n = 4r}$

In this case the matrices A and A^{-1} can be regarded as block matrices of order r with elements given by square blocks of order 4.

THEOREM 1. Matrix A^{-1} is given by the following block matrix

$$(1) \qquad A^{-1} = \begin{bmatrix} F & G & G & \cdot & \cdot & G & G \\ G^T & F & G & \cdot & \cdot & G & G \\ G^T & G^T & F & \cdot & \cdot & G & G \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ G^T & G^T & G^T & \cdot & \cdot & F & G \\ G^T & G^T & G^T & \cdot & \cdot & G^T & F \end{bmatrix}$$

where

$$F = \frac{1}{2r-1}\begin{bmatrix} 0 & 1-2r & 0 & 2r-1 \\ 1-2r & 1 & 1 & 0 \\ 0 & 1 & 1 & 1-2r \\ 2r-1 & 0 & 1-2r & 0 \end{bmatrix},$$

$$G = \frac{1}{2r-1}\begin{bmatrix} 0 & 1-2r & 0 & 2r-1 \\ 0 & 1 & 1 & 0 \\ 0 & 2r & 1 & 1-2r \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$r = 1, 2, \ldots$.

Proof. For an inductive proof, we first show that for $r = 1$ we have

$$A^{-1} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} = F .$$

Made the inductive hypothesis that (1) is true for $r = m$, verify that (1) is true also for $r = m + 1$.

Matrix A can be written

$$A = \left[\begin{array}{c|c} A_{4m} & R \\ \hline R^T & A_4 \end{array}\right]$$

where $R^T \in \mathbf{R}_{4\times 4m}$ is

$$R^T = \begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & 1 \end{bmatrix} .$$

Inverse of matrix A can then be written in the form

$$(2) \qquad A^{-1} = \begin{bmatrix} A_{4m}^{-1} + A_{4m}^{-1} R E^{-1} R^T A_{4m}^{-1} & -A_{4m}^{-1} R E^{-1} \\ -E^{-1} R^T A_{4m}^{-1} & E^{-1} \end{bmatrix} ,$$

where $(A_4 - R^T A_{4m}^{-1} R) = E$, on condition that matrix E is not singular.

Since

$$E = \frac{1}{1-2m} \begin{bmatrix} 1 & 2m & 1 & 1 \\ 2m & 1 & 2m & 1 \\ 1 & 2m & 1 & 2m \\ 1 & 1 & 2m & 1 \end{bmatrix}$$

then

$$E^{-1} = \frac{1}{2m+1} \begin{bmatrix} 0 & -1-2m & 0 & 1+2m \\ -1-2m & 1 & 1 & 0 \\ 0 & 1 & 1 & -1-2m \\ 1+2m & 0 & -1-2m & 0 \end{bmatrix}$$

which is the matrix F calculated for $r = m + 1$.

Carrying on the construction of (2) we obtain

$$-A_{4m}^{-1}RE^{-1} = \begin{bmatrix} G \\ G \\ \cdot \\ \cdot \\ G \end{bmatrix},$$

$$-E^{-1}R^{T}A_{4m}^{-1} = [G^{T}, G^{T}, \dots, G^{T}],$$

$$A_{4m}^{-1} + A_{4m}^{-1}RE^{-1}R^{T}A_{4m}^{-1} = \begin{bmatrix} F & G & \cdot & \cdot & \cdot & G \\ G^{T} & F & \cdot & \cdot & \cdot & G \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ G^{T} & G^{T} & \cdot & \cdot & \cdot & F \end{bmatrix},$$

where F and G are calculated for $r = m + 1$.

<u>$n = 4r + 2$</u>

Likewise previous case we have

THEOREM 2. Matrix A^{-1} is given by the following block matrix

$$A^{-1} = \begin{bmatrix} F & G & G & \cdot & \cdot & G & G & G_1 \\ G^{T} & F & G & \cdot & \cdot & G & G & G_1 \\ G^{T} & G^{T} & F & \cdot & \cdot & G & G & G_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ G^{T} & G^{T} & G^{T} & \cdot & \cdot & F & G & G_1 \\ G^{T} & G^{T} & G^{T} & \cdot & \cdot & G^{T} & F & G_1 \\ G_1^{T} & G_1^{T} & G_1^{T} & \cdot & \cdot & G_1^{T} & G_1^{T} & F_1 \end{bmatrix}$$

where

$$F = \frac{1}{2r+1} \begin{bmatrix} 1 & -2r & 0 & 2r+1 \\ -2r & 1 & 0 & 0 \\ 0 & 0 & 0 & -1-2r \\ 2r+1 & 0 & -1-2r & 0 \end{bmatrix},$$

$$G = \frac{1}{2r+1} \begin{bmatrix} 1 & -2r & 0 & 2r+1 \\ 1 & 1 & 0 & 0 \\ 0 & 2r+1 & 0 & -1-2r \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_1 = \frac{1}{2r+1}\begin{bmatrix} 1 & -2r \\ 1 & 1 \\ 0 & 2r+1 \\ 0 & 0 \end{bmatrix},$$

$$F_1 = \frac{1}{2r+1}\begin{bmatrix} 1 & -2r \\ -2r & 1 \end{bmatrix},\ r = 0,1,\ldots .$$

Proof. The proof is analogous to that of Theorem 1.

$n = 4r+1$

In this case we have the following

THEOREM 3. Matrix A^{-1} is given by the following block matrix

$$A^{-1} = \begin{bmatrix} F & G & G & \cdot & \cdot & G & G & g \\ G^T & F & G & \cdot & \cdot & G & G & g \\ G^T & G^T & F & \cdot & \cdot & G & G & g \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ G^T & G^T & G^T & \cdot & \cdot & F & G & g \\ G^T & G^T & G^T & \cdot & \cdot & G^T & F & g \\ g^T & g^T & g^T & \cdot & \cdot & g^T & g^T & 1-2r \end{bmatrix}$$

where

$$F = \begin{bmatrix} 1-2r & 0 & 2r & 1 \\ 0 & 0 & -1 & 0 \\ 2r & -1 & -1-2r & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 1-2r & 0 & 2r & 1 \\ 1 & 0 & -1 & 0 \\ 2r & 0 & -1-2r & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},\quad g = \begin{bmatrix} 1-2r \\ 1 \\ 2r \\ 0 \end{bmatrix},$$

$r = 0,1,2,\ldots .$

Proof. Matrix A can be written

$$A = \left[\begin{array}{c|c} A_{4r} & v \\ \hline v^T & 1 \end{array}\right].$$

Inverse of matrix A can then be written in the form

$$(3)\qquad A^{-1}=\begin{bmatrix} A_{4r}^{-1}+\alpha^{-1}A_{4r}^{-1}vv^TA_{4r}^{-1} & -\alpha^{-1}A_{4r}^{-1}v \\ -\alpha^{-1}v^TA_{4r}^{-1} & \alpha^{-1} \end{bmatrix}$$

where $\alpha=(1-v^TA_{4r}^{-1}v)=1/(1-2r)$, $r=0,1,2,\ldots$.

We have the thesis making the calculations in (3).

6. Matrix B^{-1}.

We consider the two cases of n even or odd.

$n=2k$

Matrix B can be written

$$B=\tilde{T}+ZV^T$$

where

$$\tilde{T}=\begin{bmatrix} 2 & 1 & 0 & \cdot & \cdot & 0 & 0 & 0 \\ 1 & 2 & 1 & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 1 & 2 & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 2 & 1 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & 1 & 2 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & 0 & 1 & 1 \end{bmatrix},\; Z=\begin{bmatrix} -1 & 0 \\ -1 & 0 \\ -1 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ -1 & 0 \\ -1 & 1 \end{bmatrix},\; V=\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Matrix $\tilde{T}^{-1}$ is given by (see [1])

$$\tilde{T}^{-1}=\begin{bmatrix} 1 & -1 & 1 & \cdot & \cdot & \cdot & 1 & -1 \\ -1 & 2 & -2 & \cdot & \cdot & \cdot & -2 & 2 \\ 1 & -2 & 3 & \cdot & \cdot & \cdot & 3 & -3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & -2 & 3 & \cdot & \cdot & \cdot & 2k-1 & 1-2k \\ -1 & 2 & -3 & \cdot & \cdot & \cdot & 1-2k & 2k \end{bmatrix}$$

and so we can obtain matrix B^{-1} with the use of Woodbury formula (see [2])

$$B^{-1}=\tilde{T}^{-1}-\tilde{T}^{-1}Z(I_2+V^T\tilde{T}^{-1}Z)^{-1}V^T\tilde{T}^{-1}.$$

The matrix

$$I_2 + V^T\tilde{T}^{-1}Z = \begin{bmatrix} 1-k & k \\ -k & 2k+1 \end{bmatrix}$$

is not singular $\forall k \in \mathbf{N}$; hence

$$(I_2 + V^T\tilde{T}^{-1}Z)^{-1} = \frac{1}{1+k-k^2}\begin{bmatrix} 1+2k & -k \\ k & 1-k \end{bmatrix}.$$

It follows

$$\tilde{T}^{-1}Z(I_2 + V^T\tilde{T}^{-1}Z)^{-1}V^T\tilde{T}^{-1} = \frac{1}{1+k-k^2}\begin{bmatrix} 0 & -1 \\ -1 & 2 \\ 1 & -3 \\ -2 & 4 \\ \cdot & \cdot \\ \cdot & \cdot \\ k-1 & 1-2k \\ -k & 2k \end{bmatrix}.$$

$$\cdot\begin{bmatrix} 1+2k & -k \\ k & 1-k \end{bmatrix}\cdot\begin{bmatrix} 0 & 1 & -1 & 2 & \cdot & \cdot & 1-k & k \\ -1 & 2 & -3 & 4 & \cdot & \cdot & 1-2k & 2k \end{bmatrix},$$

from which elements b'_{ij} of matrix B^{-1} are carried out.

We note that B^{-1} is symmetric and persymmetric (such as B): therefore it is enough to compute $k(k+1)$ elements b'_{ij} of B^{-1} for $1 \le i \le k$ and $i \le j \le 2k-i+1$; furthermore the elements of B^{-1}, apart the common multiplicative coefficient $\beta^{-1} = 1/(k^2-k-1)$, are given by

$$b'_{ij} = i\beta - (ij/4)(n-3) \qquad i \text{ even},\ j \text{ even};$$

$$b'_{ij} = -i\beta + i[(n-3)j-1]/4 \qquad i \text{ even},\ j \text{ odd};$$

$$b'_{ij} = -i\beta + j[(n-3)i-1]/4 \qquad i \text{ odd},\ j \text{ even};$$

$$b'_{ij} = i\beta - \tfrac{1+ij}{2}k + 3\tfrac{(i-1)(j-1)}{4} + i + j - 1 \qquad i \text{ odd},\ j \text{ odd};$$

REMARK 4. Construction of matrix B^{-1} can be also carried out (apart the coefficient β^{-1}) from the knowledge of its order and of the two first elements of the first row

$$b'_{11} = \beta - k + 1, \quad b'_{12} = -\beta + k - 2\ .$$

The other elements of the first row can be obtained as follows:

$$b'_{1j} = b'_{1,j-2} - k + 2 \quad \text{for } j \text{ odd},$$

$$b'_{1j} = b'_{1,j-2} + k - 2 \quad \text{for } j \text{ even}.$$

The elements of the second row are given by

$$b'_{22} = b'_{11} - b'_{12}\,, \qquad b'_{23} = b'_{12} - b'_{13}\,,$$

$$b'_{2j} = b'_{2,j-2} + (-1)^j(3-n) \qquad \text{for } j \geq 4.$$

The elements of the third row are given by

$$b'_{3j} = b'_{1j} - b'_{2j} \quad \text{for } j \geq 3\ .$$

From the fourth row we have

$$b'_{rj} = b'_{r-3,j} + b'_{r-2,j} - b'_{r-1,j} \quad \text{for } 4 \leq r \leq 2k \text{ and } r \leq j \leq 2k\,.$$

$\underline{n = 2k+1}$

Matrix B can be written

$$B = \tilde{T} + ZV^T$$

where, now, matrix $\tilde{T}$ is of order $2k+1$ and $Z, V \in \mathbf{R}_{(2k+1)\times 2}$.

We apply again the Woodbury formula (see [2]). The matrix

$$I_2 + V^T\tilde{T}^{-1}Z = \begin{bmatrix} -k & k+1 \\ -k-1 & 2k+2 \end{bmatrix}$$

is not singular $\forall k \in \mathbf{N}\backslash\{1\}$; hence

$$(I_2 + V^T\tilde{T}^{-1}Z)^{-1} = \frac{1}{1-k^2}\begin{bmatrix} 2+2k & -k-1 \\ k+1 & -k \end{bmatrix}.$$

It follows

$$\tilde{T}^{-1}Z(I_2 + V^T\tilde{T}^{-1}Z)^{-1}V^T\tilde{T}^{-1} = \frac{1}{1-k^2}\begin{bmatrix} -1 & 1 \\ 1 & -2 \\ -2 & 3 \\ 2 & -4 \\ \cdot & \cdot \\ \cdot & \cdot \\ k & -2k \\ -k-1 & 2k+1 \end{bmatrix}.$$

$$\cdot \begin{bmatrix} 2+2k & -k-1 \\ k+1 & -k \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 2 & -2 & \cdot & \cdot & -k & k+1 \\ 1 & -2 & 3 & -4 & \cdot & \cdot & -2k & 2k+1 \end{bmatrix}$$

from which elements b'_{ij} of matrix B^{-1} are carried out.

We note that B^{-1} is symmetric and persymmetric (such as B): therefore it is enough to compute $(k+1)^2$ elements b'_{ij} of B^{-1} for $1 \leq i \leq k$ and $i \leq j \leq 2k - i + 2$; furthermore the elements of B^{-1}, apart the common multiplicative coefficient $\beta^{-1} = 1/(1-k^2)$, are given by

$$b'_{ij} = i\beta - (ij/2)(1-k) \qquad i \text{ even},\ j \text{ even};$$

$$b'_{ij} = -i\beta - (ij/2)(k-1) \qquad \begin{cases} i \text{ even},\ j \text{ odd}; \\ i \text{ odd},\ j \text{ even}; \end{cases}$$

$$b'_{ij} = i\beta - (ij/2)(1-k) + (k+1)/2 \qquad i \text{ odd},\ j \text{ odd}.$$

REMARK 5. Construction of matrix B^{-1} can be also carried out (apart the coefficient β^{-1}) from the knowledge of its order and of the two first elements of the first row

$$b'_{11} = \beta + k, \quad b'_{12} = -\beta - k + 1\ .$$

The other elements of the first row can be obtained as follows:

$$b'_{1j} = b'_{1,j-2} + k - 1 \quad \text{for } j \text{ odd},$$

$$b'_{1j} = b'_{1,j-2} - k + 1 \quad \text{for } j \text{ even}.$$

The elements of the second row are given by

$$b'_{2j} = b'_{1,j-1} - b'_{1j} - 1 \quad \text{for } j \geq 2 \text{ and } j \text{ even},$$

$$b'_{2j} = b'_{1,j-1} - b'_{1j} + 1 \quad \text{for } j > 2 \text{ and } j \text{ odd}.$$

The elements of the third row are given by

$$b'_{3j} = b'_{1j} - b'_{2j} \quad \text{for } j \geq 3\ .$$

From the fourth row we have

$$b'_{rj} = b'_{r-3,j} + b'_{r-2,j} - b'_{r-1,j} \quad \text{for } 4 \leq r \leq 2k+1 \text{ and } r \leq j \leq 2k+1\ .$$

7. Haenkel matrices.

From previous results can be carried out analogous results for the two following classes of Haenkel matrices

$$\hat{A} = \begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 & 0 & 1 \\ 1 & 1 & 1 & \cdot & \cdot & 0 & 1 & 0 \\ 1 & 1 & 1 & \cdot & \cdot & 1 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 1 & \cdot & \cdot & 1 & 1 & 1 \\ 0 & 1 & 0 & \cdot & \cdot & 1 & 1 & 1 \\ 1 & 0 & 1 & \cdot & \cdot & 1 & 1 & 1 \end{bmatrix}$$

and

$$\hat{B} = \begin{bmatrix} -1 & -1 & -1 & \cdot & \cdot & -1 & 0 & 1 \\ -1 & -1 & -1 & \cdot & \cdot & 0 & 1 & 0 \\ -1 & -1 & -1 & \cdot & \cdot & 1 & 0 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 0 & 1 & \cdot & \cdot & -1 & -1 & -1 \\ 0 & 1 & 0 & \cdot & \cdot & -1 & -1 & -1 \\ 1 & 0 & -1 & \cdot & \cdot & -1 & -1 & -1 \end{bmatrix}.$$

In fact as $\hat{A} = AJ$ we have $\hat{A}^{-1} = JA^{-1}$ and, likewise, from $\hat{B} = BJ$ we obtain $\hat{B}^{-1} = JB^{-1}$.

Also for these two classes of Haenkel matrices can be given $[n/2]$ eigenvalues and intervals of inclusion for the others: in fact transforming matrix $\hat{A}$, for $n = 2k$, with the use of S_1, and for $n = 2k + 1$, with the use of S_2, we obtain, respectively, the two block diagonal matrices

$$S_1^{-1}\hat{A}S_1 = \begin{bmatrix} M + NJ & 0 \\ 0 & -(M - JN) \end{bmatrix},$$

$$S_2^{-1}\hat{A}S_2 = \left[\begin{array}{cc|c} M + U & \sqrt{2}v & 0 \\ \sqrt{2}v^T & 1 & 0 \\ \hline 0 & 0 & -(M - U) \end{array}\right].$$

Therefore $[n/2]$ eigenvalues are opposite to those of matrix A (these are known exactly) while for the others exist opportune intervals of inclusion.

This remark can be done also for matrix $\hat{B}$ and we obtain, for $n = 2k$ and $n = 2k + 1$, respectively,

$$S_1^{-1}\hat{B}S_1 = \begin{bmatrix} C - NJ & 0 \\ 0 & -(C + JN) \end{bmatrix},$$

$$S_2^{-1}\hat{B}S_2 = \left[\begin{array}{cc|c} C - U & -\sqrt{2}v & 0 \\ -\sqrt{2}v^T & 1 & 0 \\ \hline 0 & 0 & -(C + U) \end{array}\right]:$$

for these analogous remarks made for matrix $\hat{A}$ are available.

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