

ON A PROCEDURE FOR THE SIMULTANEOUS DETERMINATION OF ALL ZEROS OF A POLYNOMIAL (*)

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SOMMARIO. - *In questo lavoro si considera una procedura d'iterazione per la simultanea determinazione di tutti gli zeri di un polinomio nel caso che essi siano reali e distinti. Si ottengono condizioni sufficienti per la convergenza di una tale procedura.*

SUMMARY. - *We consider in this paper an iteration procedure for the simultaneous determination of all zeros of a polynomial in the case when they are real and distinct. The sufficient conditions for the convergence of that procedure are obtained.*

Let us consider the system of equations

$$(1) \quad x_i = f_i(x_1, x_2, \dots, x_n) \quad (i = 1, 2, \dots, n)$$

where the f_i and their derivatives $\frac{\partial f_i}{\partial x_j}$ ($i, j = 1, 2, \dots, n$) are real and continuous functions for $A_i \leq x_i \leq B_i$, where A_i and B_i are given numbers. Let x'_1, x'_2, \dots, x'_n be an isolated solution of (1), with $A_i \leq x'_i \leq B_i$. As is known, the quoted isolated solution of the system (1) can be obtained by the iteration method. In order to obtain it we start with a given initial approximate solution $x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}$, with $A_i \leq x_i^{(0)} \leq B_i$, and form the iteration sequences

$$(2) \quad x_i^{(k+1)} = f_i(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}) \quad (i = 1, 2, \dots, n; k = 0, 1, 2, \dots).$$

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If we set

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad f(X) = \begin{bmatrix} f_1(X) \\ f_2(X) \\ \vdots \\ f_n(X) \end{bmatrix},$$

then the system (1) and the iteration sequences (2) can be written as

$$(1') \quad X = f(X)$$

and

$$(2') \quad X^{(k+1)} = f(X^{(k)}) \quad (k = 0, 1, 2, \dots).$$

For the derivative $f'(X)$ we now have

$$f'(X) = \left[\frac{\partial f_i}{\partial x_j} \right].$$

The domain of X , $f(X)$ and $f'(X)$ is $D\{A_i \leq x_i \leq B_i\}$, where A_i and B_i are given real numbers.

Here we use the norm notation

$$\|X\|_m = \max_i |x_i|.$$

With respect to domain D we shall introduce the norm

$$(3) \quad \|f'(X)\|_I = \max_{X \in D} \|f'(X)\|_m,$$

where

$$(4) \quad \|f'(X)\|_m = \max_i \sum_{j=1}^n \left| \frac{\partial f_i(X)}{\partial x_j} \right|.$$

In our paper we shall use the following theorem related to the convergence of the iterative procedure (2'), that is (2), and which may be found in [1].

THEOREM 1. *Let the functions $f(X)$ and $f'(X)$ be continuous in the domain D , and, in D , let the inequality*

$$(5) \quad \|f'(X)\|_I \leq q < 1 ,$$

where q is a constant, hold true.

If the successive approximations

$$(6) \quad X^{(k+1)} = f(X^{(k)}) \quad (k = 0, 1, 2, \dots)$$

lie in D , then the iteration process (6) converges and the limiting vector

$$X' = \lim_{k \rightarrow \infty} X^{(k)}$$

is the sole solution of system (1) in domain D .

COROLLARY 1. *The iteration process (6) converges if*

$$(7) \quad \sum_{j=1}^n \left| \frac{\partial f_i(X)}{\partial x_j} \right| \leq q_i < 1 \quad (i = 1, 2, \dots, n)$$

when $X \in D$.

Obviously, from the system of inequalities (7) follows condition (5) of Theorem 1.

NOTE 1. For the approximation of $X^{(k)}$ the following estimate holds:

$$(8) \quad \|X' - X^{(k)}\|_m \leq \frac{q^k}{1-q} \|X^{(1)} - X^{(0)}\|_m \quad (k = 1, 2, \dots) ,$$

where $X^{(1)} = f(X^{(0)})$.

NOTE 1'. The convergence of the iterative process (2) (or (2')) in the general case is linear.

Here a proof of Note 1' will be given.

If the iterative process (2') converges to the solution X' of the equation (1'), then

$$(9) \quad X' = f(X') .$$

Let $X^{(k)}$ be an approximate solution of equation (1') obtained by the process (2') for k large enough. Setting $X' = X^{(k)} + (X' - X^{(k)})$ and using Taylor's formula we obtain

$$(10) \quad f(X') = f(X^{(k)}) + f'(\bar{X})(X' - X^{(k)})$$

whence $\bar{X} = X^{(k)} + \theta(X' - X^{(k)})$, $0 < \theta < 1$.

Keeping in mind (2') and (9), we obtain from (10)

$$X' - X^{(k+1)} = f'(\bar{X})(X' - X^{(k)}),$$

whence

$$(11) \quad \|X' - X^{(k+1)}\|_m \leq \max_{X \in D} \|f'(X)\|_m \|X' - X^{(k)}\|_m.$$

Because of (3), (4) and (5) we see that (11) is reduced to

$$(12) \quad \|X' - X^{(k+1)}\|_m \leq q \|X' - X^{(k)}\|_m.$$

From (12) we deduce that the convergence of the iterative process (2') (or (2)) is linear, that is of the order 1.

1. Consider a real polynomial

$$(13) \quad P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \quad (a_n \neq 0, n \geq 3)$$

with all zeros x_1, x_2, \dots, x_n real and distinct.

Let us set then

$$d = \min_{i \neq j} |x_i - x_j| \quad (i, j = 1, 2, \dots, n).$$

For d one usually takes an estimate of the form

$$d \geq m \quad (m > 0),$$

(see for example [2], also [3]).

When discussing a real polynomial whose all zeros x_i are real and distinct, which is our case, the lower bound for d can be also determined if

all the zeros of the polynomial (13) are sufficiently separated using corresponding intervals.

Let

$$(14) \quad [b_i, d_i], \quad (i = 1, 2, \dots, n), \quad b_1 < d_1 < b_2 < d_2 < \dots < b_n < d_n$$

denote intervals which contain the respective zeros x_i of the polynomial (13). Then

$$(d_i, b_{i+1}) \quad (i = 1, 2, \dots, n-1)$$

are intervals in which the polynomial (13) has no zeros. It is obvious that

$$d \geq \min_i (b_{i+1} - d_i) \quad (i = 1, 2, \dots, n-1).$$

We shall write polynomial (13) in the form

$$a_n \prod_{j=1}^n (x - x_j) = P(x), \quad \text{whence} \quad \frac{1}{x - x_i} + \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{x - x_j} = \frac{P'(x)}{P(x)} \quad \text{and}$$

$$(15) \quad \frac{1}{(x - x_i)^3} + \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{(x - x_j)^3} = Q(x) \quad (i = 1, 2, \dots, n)$$

where

$$(16) \quad Q(x) = \frac{P'''(x)P^2(x) - 3P''(x)P'(x)P(x) + 2(P'(x))^3}{2P^3(x)}.$$

We shall take respectively from each of the intervals (14) one point. Let these points be c_i ($i = 1, 2, \dots, n$), no one of them being a zero of (13). We then have from (15), for $x = c_i$,

$$(17) \quad \frac{1}{(c_i - x_i)^3} + \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{(c_i - x_j)^3} = Q(c_i) \quad (i = 1, 2, \dots, n).$$

We shall consider (17) as a system of n equations in n unknowns x_1, x_2, \dots, x_n which can be written in the form

$$(18) \quad x_i = c_i - \left[Q(c_i) - \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{(c_i - x_j)^3} \right]^{-\frac{1}{3}} \quad (i = 1, 2, \dots, n).$$

The system (18) is of the form (1). We shall apply the method of iteration for its solution. Let $x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}$ be initial approximate values of zeros of the polynomial (13), with $x_i^{(0)} \in [b_i, d_i], (i = 1, 2, \dots, n)$. Applying the iteration procedure (2) to the system (18) we obtain

$$(19) \quad x_i^{(k+1)} = c_i - \left[Q(c_i) - \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{(c_i - x_j^{(k)})^3} \right]^{-\frac{1}{3}},$$

$(i = 1, 2, \dots, n; k = 0, 1, 2, \dots)$.

We shall demonstrate the following theorem, related to the convergence of iteration sequences (19).

THEOREM 2. *Let the polynomial (13) have all its zeros $x_i (i = 1, 2, \dots, n)$ real and distinct contained in the intervals (14), respectively. If in every interval (14) we choose arbitrarily c_i and $x_i^{(0)}$ and if then all the approximations $x_i^{(k+1)}$ ($k = 0, 1, 2, \dots$) obtained by the procedure (19) remain in the intervals (14), then the iteration sequences (19) converge respectively to the zeros x_i of the polynomial (13), when*

$$(20) \quad \max_i (d_i - b_i) \leq \frac{d}{s},$$

where

$$d = \min_{i \neq j} |x_i - x_j|,$$

s is a number greater than 2 for which

$$(21) \quad \left[\left(\frac{s}{s-1} \right)^4 + \frac{\pi^4}{45} \right] \cdot \frac{1}{s^4} \leq q < 1$$

and q is a constant.

Proof. The system (18) is of the form (1). Denoting its right sides by $f_i (i = 1, 2, \dots, n)$ respectively, we shall have

$$(22) \quad \frac{\partial f_i}{\partial f_j} = \begin{cases} 0 & \text{for } j = i \\ -(c_i - x_i)^{-4} & \left[Q(c_i) - \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{(c_i - x_j)^3} \right]^{-\frac{4}{3}} \end{cases} \quad \text{for } j \neq i.$$

Substituting in the relation (22) the quantities $Q(c_i)$ by the corresponding values from (17) we shall obtain

$$(23) \quad \frac{\partial f_i}{\partial f_j} = \begin{cases} 0 & \text{for } j = i \\ \frac{-(c_i - x_i)^4}{(c_i - x_j)^4} & \text{for } j \neq i \end{cases} .$$

By (23), the left sides in (7) reduce to

$$(24) \quad \sum_{\substack{j=1 \\ j \neq i}}^n \frac{|c_i - x_i|^4}{|c_i - x_j|^4} \quad (i = 1, 2, \dots, n) .$$

By hypothesis, the zeros x_1, x_2, \dots, x_n of (13) are real and distinct and the minimal distance between neighbouring zeros is $d > 0$. Let zeros x_1, x_2, \dots, x_n be in order

$$x_1 < x_2 < \dots < x_n .$$

If $x_i, c_i, x_i^{(0)} \in [b_i, d_i]$, then, having in mind (20)

$$(25) \quad |c_i - x_i| \leq \max_i (d_i - b_i) \leq \frac{d}{s}$$

and

$$(25') \quad |x_i^{(0)} - x_i| \leq \max_i (d_i - b_i) \leq \frac{d}{s} \quad (s > 2), \quad (i = 1, 2, \dots, n) .$$

If $c_i < x_i$, because of (25), we now have

$$(26) \quad \begin{aligned} |c_i - x_j| &> (j - i)d \text{ for } j - i > 0, \\ |c_i - x_j| &> (i - j - 1)d \text{ for } i - j - 1 > 0, \\ |c_i - x_j| &\geq \frac{s-1}{s}d \text{ for } i - j - 1 = 0 \quad (i, j = 1, 2, \dots, n) . \end{aligned}$$

If $c_i > x_i$, then, having in mind (25)

$$(27) \quad \begin{aligned} |c_i - x_j| &> (j - i - 1)d \text{ for } j - i - 1 > 0, \\ |c_i - x_j| &\geq \frac{s-1}{s}d \text{ for } j - i - 1 = 0, \\ |c_i - x_j| &> (i - j)d \text{ for } i - j > 0 \quad (i, j = 1, 2, \dots, n) . \end{aligned}$$

From (26) and (27) one obtains

$$(28) \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{|c_i - x_j|^4} < \frac{1}{d^4} \left[\left(\frac{s}{s-1} \right)^4 + 2 \left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots + \frac{1}{\left(\frac{n-2}{2}\right)^4} \right) \right], \quad (i = 1, 2, \dots, n)$$

for n even ≥ 4 , and

$$(29) \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{|c_i - x_j|^4} < \frac{1}{d^4} \left[\left(\frac{s}{s-1} \right)^4 + 2 \left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots + \frac{1}{\left(\frac{n-1}{2}\right)^4} \right) \right], \quad (i = 1, 2, \dots, n)$$

for n odd ≥ 3 .

Since

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90},$$

relations (28) and (29) reduce to

$$(30) \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{|c_i - x_j|^4} < \frac{1}{d^4} \left[\left(\frac{s}{s-1} \right)^4 + \frac{\pi^4}{45} \right].$$

Multiplying (30) by $|c_i - x_i|^4$, one obtains, keeping in mind (25)

$$(31) \sum_{\substack{j=1 \\ j \neq i}}^n \frac{|c_i - x_i|^4}{|c_i - x_j|^4} < \left[\left(\frac{s}{s-1} \right)^4 + \frac{\pi^4}{45} \right] \cdot \frac{1}{s^4} \quad (i = 1, 2, \dots, n).$$

Because of (21) we see that (31) reduces to

$$\sum_{\substack{j=1 \\ j \neq i}}^n \frac{|c_i - x_i|^4}{|c_i - x_j|^4} \leq q < 1 \quad (i = 1, 2, \dots, n),$$

which by (23) and (24) represents the conditions (7). The condition (20) implies that (7) is satisfied. In view of Corollary 1 and Theorem 1, the preceding fact proves Theorem 2.

NOTE 2. For the approximations of $x_i^{(k)}$ the following estimates hold (32)

$$|x_i^{(k)} - x_i| \leq \frac{q^k}{1-q} \max_i |x_i^{(1)} - x_i^{(0)}| \quad (i = 1, 2, \dots, n; k = 1, 2, \dots).$$

NOTE 2'. The convergence of the iterative process (19) is linear.

COROLLARY 2. *The iteration process (19) converges if*

$$(32') \quad \max_i (d_i - b_i) \leq 0.4915563988 d.$$

The estimate (32) follows from the estimate (8), while the conclusion in Note 2' follows from the conclusion of Note 1'.

Set

$$(33) \quad F(s) = \left[\left(\frac{s}{s-1} \right)^4 + \frac{\pi^4}{45} \right] \cdot \frac{1}{s^4}.$$

For $s \in (0, 1)$ we have $F(s) > 2$. For $s \in (1, \infty)$ the function $F(s)$ is continuous and monotonically decreasing, where $F(s) \rightarrow \infty$ when $s \rightarrow 1$ and $F(s) \rightarrow 0$ when $s \rightarrow \infty$. Thus there is only one value $s' \in (1, \infty)$ for which $F(s') = 1$, which means that s' is the positive root of the equation

$$(34) \quad \left[\left(\frac{s}{s-1} \right)^4 + \frac{\pi^4}{45} \right] \cdot \frac{1}{s^4} = 1.$$

It is not hard to establish that for s' we have the bounds

$$2.034354556 < s' < 2.034354557.$$

Besides that, for $s > s'$ one has

$$(35) \quad F(s) = q < F(s') < 1.$$

In view of (33) the relation (35) represents the condition (21), which means that (20) holds for $s > s'$.

REMARK. We saw that the condition (20) is fulfilled for $s > s'$, where s' is the positive root of the equation (34). Therefore in (20) one may take $s = 2.034354557$. In that case (20) reduces to

$$(36) \quad \max_i (d_i - b_i) \leq 0.4915563988 d,$$

so that Corollary 2 is proved.

For given $s > s'$, in view of (21), one has

$$(37) \quad q = \left[\left(\frac{s}{s-1} \right)^4 + \frac{\pi^4}{45} \right] \cdot \frac{1}{s^4} < 1,$$

and according to (20) one should have

$$(38) \quad \max_i (d_i - b_i) \leq \frac{d}{s}.$$

The constant q obtained in this way serves in (32) for the valuation of the approximations $x_i^{(k)}$.

From (37) we may deduce that q quickly decreases as s gets larger, which in view of (38) means that q decreases quickly when $\max_i (d_i - b_i)$ decreases.

For a given $q < 1$ one can find s from (37), and then from (38) determine how large $\max_i (d_i - b_i)$ should be. In the following table we give some cases:

q	s	$\max_i (d_i - b_i)$	q	s	$\max_i (d_i - b_i)$
0.5	2.246	$\leq 0.4452 d$	0.001	7.900	$\leq 0.1265 d$
0.1	2.940	$\leq 0.3401 d$	0.0001	13.698	$\leq 0.0730 d$
0.01	4.677	$\leq 0.2138 d$	0.00001	24.390	$\leq 0.0410 d$

In practice $\max_i (d_i - b_i)$ should be determined so that (36) holds. Then from (38) one takes $s = \frac{d}{\max_i (d_i - b_i)}$. With this value of s one determines the constant q from (37).

As mentioned, for the quantity d one usually gives the bound of the form $d \geq m$ ($m > 0$), so that the condition (20) will be satisfied for

$$(39) \quad \max_i (d_i - b_i) \leq \frac{m}{s},$$

while the condition (36) will be satisfied when

$$(40) \quad \max_i (d_i - b_i) \leq 0.4915563988 m.$$

In practice instead of d one most often operates with the quantity m .

In iterative procedures for simultaneous determination of all zeros x_i of a polynomial (when the zeros are all distinct) the initial values $x_i^{(0)}$ must be chosen so that

$$|x_i^{(0)} - x_i| \leq h \quad (i = 1, 2, \dots, n).$$

To prove the convergence of these iterative procedures, the constant h , in the general case must satisfy the condition

$$h < \frac{d}{2}.$$

In some iterative procedures one requires that h be much smaller than $\frac{d}{2}$.

We mention here the procedure

$$(41) \quad x_i^{(k+1)} = x_i^{(k)} - \frac{P(x_i^{(k)})}{\prod_{\substack{j=1 \\ j \neq i}}^n (x_i^{(k)} - x_j^{(k)})} \quad (i = 1, 2, \dots, n; k = 0, 1, 2, \dots)$$

which is used more than 20 years. The appearance of the procedure (41) is linked with the papers [4] and [5].

For the iterative procedure (41), K. Dočev [4] proved the following proposition:

If $d = \min_{i \neq j} |x_i - x_j|$ and

$$(41') \quad |x_i^{(0)} - x_i| < h \quad (i = 1, 2, \dots, n)$$

where

$$(42) \quad h = d \cdot \frac{(1+q)^{1/(n-1)}}{2(1+q)^{1/(n-1)} - 1} \quad (0 < q < 1),$$

then the iterative procedure (41) is convergent, where

$$|x_i^{(k)} - x_i| \leq hq^{2^k-1} \quad (i = 1, 2, \dots, n; k = 0, 1, 2, \dots).$$

By the procedure (41), we deduce from (42) that the quantity h must be much smaller than $\frac{d}{2}$ and it decreases with the increase of the degree n of the polynomial. Thus for $n = 3$ we must have $h < 0.2265409197 d$, for $n = 6$ we must have $h < 0.1146128658 d$, and for $n = 10$ we must have $h < 0.0690099084 d$. A similar situation arises with some interval methods for simultaneous determination of all zeros of a polynomial (see [6]).

Good aspects of the procedure (19) are in the following:

- 1) one can take wider intervals $[b_i, d_i]$ containing initial values $x_i^{(0)}$,
- 2) the widths of the intervals $[b_i, d_i]$, depending on (20), do not depend on the degree n of the polynomial,
- 3) a relatively small decrease of the intervals $[b_i, d_i]$ enables one to arrive at the approximations $x_i^{(k)}$ with a satisfying precision,
- 4) the calculation of the values $x_i^{(k+1)}$ are not complicated, since the quantities c_i and $Q(c_i)$ are not changed.

2. EXAMPLE. The procedure (19) will be applied to the Legendre polynomial of degree 6

$$(43) \quad P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5).$$

It is known that the Legendre polynomial of degree n

$$P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n} (x^2 - 1)^n$$

has all zeros real and different and that they all lie in the interval $(-1, 1)$.

The zeros of the polynomial (43) are the zeros of the polynomial

$$(44) \quad P(x) = 231x^6 - 315x^4 + 105x^2 - 5$$

and they are symmetrically distributed with respect to the point 0. Since $P(-0.95) > 0$, $P(-0.92) < 0$; $P(-0.67) < 0$, $P(-0.64) > 0$; $P(-0.25) > 0$, $P(-0.21) < 0$; $P(0.21) < 0$, $P(0.25) > 0$; $P(0.64) > 0$, $P(0.67) < 0$; $P(0.92) < 0$, $P(0.95) > 0$; then in every interval $[b_i, d_i]$:

$$(45) \quad [-0.95, -0.92], [-0.67, -0.64], [-0.25, -0.21], \\ [0.21, 0.25], [0.64, 0.67], [0.92, 0.95]$$

lies a zero of the polynomial (44). Here

$$(46) \quad \max_i (d_i - b_i) = 0.04$$

and

$$(47) \quad \min_i (b_{i+1} - d_i) = 0.25 = m \leq d.$$

According to (47) we have

$$0.4915563988 \ m = 0.1228890997$$

which in view of (46) means that

$$(48) \quad \max_i (d_i - b_i) < 0.4915563988 \ m.$$

Taking into account (47) we obtain from (48)

$$\max_i (d_i - b_i) < 0.4915563988 \ d,$$

so that also the condition (32') is fulfilled. According to (39), we now have $s = \frac{m}{\max_i (d_i - b_i)} = 6.25$. For $s = 6.25$ from (37) we obtain

$$q < 0.002735.$$

If for quantities $c_i \in [b_i, d_i]$ and $x_i^{(0)} \in [b_i, d_i]$ we take for example:

$$c_1 = -0.94, c_2 = -0.63, c_3 = -0.23, c_4 = 0.22, c_5 = 0.65, c_6 = 0.93,$$

$$x_1^{(0)} = -0.93, x_2^{(0)} = -0.65, x_3^{(0)} = -0.24,$$

$$x_4^{(0)} = 0.24, x_5^{(0)} = 0.66, x_6^{(0)} = 0.94,$$

then by applying the procedure (19), taking into account (16), we set approximations:

$$x_1^{(1)} = -0.9324695197, x_1^{(2)} = -0.9324695142, x_1^{(3)} = -0.9324695142$$

$$x_2^{(1)} = -0.6612096179, x_2^{(2)} = -0.6612093865, x_2^{(3)} = -0.6612093865$$

$$x_3^{(1)} = -0.2386191882, x_3^{(2)} = -0.2386191861, x_3^{(3)} = -0.2386191861$$

$$x_4^{(1)} = 0.2386191878, x_4^{(2)} = 0.2386191861, x_4^{(3)} = 0.2386191861$$

$$x_5^{(1)} = 0.6612093679, x_5^{(2)} = 0.6612093865, x_5^{(3)} = 0.6612093865$$

$$x_6^{(1)} = 0.9324695142, x_6^{(2)} = 0.9324695142, x_6^{(3)} = 0.9324695142$$

According to (32), in our case

$$|x_i^{(2)} - x_i| < 0.0000000841 \quad (i = 1, 2, \dots, 6).$$

For the application of the procedure (41), according to (41') and (42) we must have $|x_i^{(0)} - x_i| < 0.1146128658$ $m = 0.0286532165$, that is $\max_i (d_i - b_i) < 0.0286532165$, which is not fulfilled here. This means that in applying the procedure (41) the initial values $x_i^{(0)}$ must be taken from smaller intervals than the intervals (45).

When dealing with a complex or real polynomial $P(z)$ of degree n , whose all zeros z_1, z_2, \dots, z_n are distinct, then we have the following proposition:

If \bar{z}_i is an approximate value of a zero z_i of a polynomial $P(z)$ and if

$$(49) \quad \left| \frac{nP(\bar{z}_i)}{P'(\bar{z}_i)} \right| < \frac{m}{2} \leq \frac{d}{2},$$

then

$$(50) \quad |\bar{z}_i - z_i| < \frac{|P(\bar{z}_i)|}{|P'(\bar{z}_i)| - \frac{2n-2}{m}|P(\bar{z}_i)|}$$

independently of the way we obtained the approximate value \bar{z}_i . The proposition (50) may be found in [7].

In our case, for $\bar{z}_i = x_i^{(2)}$ ($i = 1, 2, \dots, 6$), the conditions (49) are fulfilled. According to (50) we have

$$|x_i^{(2)} - x_i| < 0.000000000036 \quad (i = 1, 2, \dots, 6) .$$

Let us state at the end that there exist several procedures for the simultaneous determination of all zeros of a polynomial. Some of them may be found in [8] and [9].

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