EXISTENCE OF SOLUTIONS TO THE PROBLEM OF NONEQUILIBRIUM PHASE CHANGE (*)

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SOMMARIO. - *In questo lavoro si considera il seguente problema di "nonequilibrium phase-change":*

$$\begin{cases} c\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0,\ 0 < x < 1,\ 0 < t < T, \\ u(0,t) = h(t),\ u(1,t) = h(t), \\ u(x,0) = u(x),\ s(0) = s_0, \\ k\left[\frac{\partial u(s(t)+,t)}{\partial x} - \frac{\partial u(s(t)-,t)}{\partial x}\right] = L\dot{s}(t) = Lg(u(s(t),t)) \end{cases}$$

e si dimostra l'esistenza di sue soluzioni deboli o classiche sotto certe condizioni.

SUMMARY. - *In this paper we consider the following problem of nonequilibrium phase-change:*

$$\begin{cases} c\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0,\ 0 < x < 1,\ 0 < t < T, \\ u(0,t) = h(t),\ u(1,t) = h(t), \\ u(x,0) = u(x),\ s(0) = s_0, \\ k\left[\frac{\partial u(s(t)+,t)}{\partial x} - \frac{\partial u(s(t)-,t)}{\partial x}\right] = L\dot{s}(t) = Lg(u(s(t),t)) \end{cases}$$

and prove the existence of its weak or classical solutions under some conditions.

1. Introduction.

The classical, one dimension and two phase Stefan problem (free boundary problem) is to find the couple functions $\{u(x,t), s(t)\}$, such

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that they satisfy the following system:

$$\begin{cases} c\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0\,, & \text{in } Q_T\backslash S\,, \quad (1.1)\\ u(0,t) = h_1(t),\ u(1,t) = h_2(t)\,, & 0 < t < T\,, \quad (1.2)\\ u(x,0) = u_0(x)\,, & 0 < x < 1\,, \quad (1.3)\\ s(0) = s_0\,, & 0 < s_0 < 1\,, (1.4)\\ k\left[\frac{\partial u(s(t)+,t)}{\partial x} - \frac{\partial u(s(t)-,t)}{\partial x}\right] = L\dot{s}(t)\,, & 0 < t < T\,, \quad (1.5)\\ u(s(t),t) = 0\,, & 0 < t < T\,. \quad (1.6) \end{cases}$$

where $Q_T = (0,1) \times (0,T)$, $S = \{(s(t),t), t \in (0,T)\}$, $u(x,t)$ denotes the difference between the actual temperature and the equilibrium phase transition temperature, $0 < x < s(t)$ corresponds to the liquid and $s(t) < x < 1$ to the solid, c is the product of the specific heat and the density, k the conductivity, L the product of the latent heat of phase transition and the density, and s_0 and T are fixed constants. In problem (1.1)–(1.6), supercooled or superheated state is excluded in general. The problems for the existence and uniqueness and so on of solutions are quite complete (see proceedings [1] and [2]). But for so termed nonequilibrium problems, we have to consider the supercooled or superheated phenomena. In this case, we replace the condition (1.6) by the following:

$$\dot{s}(t) = g(u(s(t),t))\,. \qquad (1.7)$$

We call the Problem (1.1)–(1.5) and (1.7) a nonequilibrium one (NP). In [3], A. Crowley presented a variety of g form arising in a number of physical situations and investigated the properties of travelling wave solutions only. In [4], A. Visitin proved that the problem (NP) has at least a weak solution only provided g is almost linear. For general g, up to now, we have not seen any results. In this paper, we consider the general g, especially including all algebraic polynomial, to prove the existence theorem under some assumptions. In section 2 we state the main results. The proofs are in section 3. Some remarks in section 4.

2. Main results.

Assuming

(A1) $\qquad g(y) \in C^1(R)$ and $|g(y)| \le C(|y|^m + 1)\,,$

$$|g'(y)| \leq C(|y|^{m-1} + 1) ,$$

where C and $m \geq 1$ are positive constants,

(A2) $$h_i(t) \in C^{\alpha_1}[0,T],\ i = 1,2,\ 0 < \alpha_1 < 1 ,$$

$$u_0(x) \in C^{\alpha_2}[0,1],\ 0 < \alpha_2 < 1 \text{ and}$$

$$h_1(0) = u_0, h_2(1) = u_0(1) ,$$

then we have the following:

THEOREM 1. (local existence theorem). *Under the assumptions* (A1) *and* (A2), *then there exists a* $t_1 \in (0,T]$ *such that the problem* (NP) *has at least a weak solution*

$$\{u(x,t), s(t)\} \text{ in } Q_{t_1} .$$

Here the definition of weak solutions is as follows:

$s(t) \in C^1[0,t_1]$ satisfying (1.4), $u(x,t) \in C^0(\bar{Q}_{t_1}) \cap V_1^{1,0}(Q_{t_1})$ satisfying (1.2) and (1.3), and for any $t \in (0,t_1)$ the following equality holds.

$$\int_{Q_t} \{-(u\varphi_t + [ku_k + Lg(u(s(t),t))H(s(t) - x)]\varphi_x\}\, dxdt$$

$$-\int_0^1 (u_0(x)\varphi(x,0)\ dx = 0, \text{for all } \varphi \in \overset{\circ}{w}{}_2^{1,1}(Q_t) \text{ with } \varphi(x,t) = 0, \tag{2.1}$$

where

$$H(y) = 1 \text{ as } y > 0 \text{ and } 0 \text{ as } y \leq 0 . \tag{2.2}$$

If we claim slightly more smoothness of $h_i(t)$ and $u_0(x)$, i.e. assuming

(A2)′ $$h_i(t) \in C^{\alpha_1}[0,T] , 1/2 < \alpha_1 < 1,\ i = 1,2 .$$

$$u_0(x) \in C^{\alpha_2}[0,s_0] \cap C^{\alpha_2}[s_0,1],\ 2 < \alpha_2 < 3,\ u_0(s_0+) = u_0(s_0-) ,$$

$$(k/L)[u_{0x}(s_0+) - u_{0x}(s_0-)] = g(u_0(s_0)) ,$$

$$h_1(0) = u_0(0) \text{ and } h_2(1) = u_0(1) ,$$

then we have

THEOREM 1′. *Under the assumptions* (A1) *and* (A2)′, *then the weak solution in Theorem* 1 *is also classical.*

Here the definition of classical solution is the following: $s(t) \in C^1[0, t_1]$, $u(x,t) \in C^{2,1}(Q_{1t_1} Q_{2t_1}) \cap C^0(\bar{Q}_{t_1})$ and $u_x(x,t) \in C^0(\bar{Q}_{1t_1}) \cap C^0(\bar{Q}_{2t_1})$, and they satisfy (1.1)–(1.5) and (1.7) classically, where $Q_{1t_1} = (0, s(t)) \times (0, t_1)$ and $Q_{2t_1} = (s(t), 1) \times (0, t_1)$.

If further assuming for g

(A1)′ $\quad |g(y)| \le C(|y| + 1)$ or $yg(y) \ge 0$, for all $y \in \mathbf{R}$,

then we have the existence theorem for the global solutions.

THEOREM 2. *Under the assumptions* (A1)′ *and* (A2), *the problem* (NP) *has at least a weak solution either in Q_T or in Q_{T_1} which is a maximum domain for the existence of solutions and if $T_1 < T$, then $s(T_1) = 0$ or $s(T_1) = 1$.*

Corresponding to Theorem 1′, we have

THEOREM 2′. *Under the assumptions* (A1)′ *and* (A2)′, *then the weak solution in Theorem* 2 *is also classical.*

3. Proofs.

In order to prove Theorem 1, we use Schauder's fixed point theorem. So we select the closed convex subset

$$E = \{b(t) \in L^p[0, t_1] : \|b(t)\|_{L^p(0,t_1)} \le K_1\} \tag{3.1}$$

in Banach space $L^p[0, t_1]$, where t_1 is to be determined a small positive constant and

$$p = 4m . \tag{3.2}$$

If $b \in E$ and g satisfies the condition (A1), then for any $t \in (0, t_1)$ we have

$$\int_0^t |g(b(y))|^4 dy \le \int_0^t C^4(|b(y)|^m + 1\}^4 dy \le K_2 \ , \qquad (3.3)$$

where K_2 is a constant depending only on C and K_1, and considering K_2 as function of t, we get that it is increasing with respect to t.

Throughout this paper, we denote the generic constant by K depending only on the given data, K_1 and K_2, and use the notations as the same as that in [6]. Without loss of generality, we assume $c = k = L = 1$.

For given $b \in E$, we consider the following auxiliary problem (AP):

$$(\mathrm{AP}) \begin{cases} \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(u_x + g(b(t))H(s(t)) - x)) = 0 & \text{in } Q_{t_1} \ , \qquad (3.4) \\ u(0,t) = h_1(t), \ u(1,t) = h_2(t) \ , & 0 < t < t_1 \ , (3.5) \\ u(0,x) = u_0(x) \ , & 0 < x < 1 \ , (3.6) \\ s(t) = s_0 + \int_0^t g(b(y))dy \ . & (3.7) \end{cases}$$

Choosing t_1 so small such that $0 < s(t) < 1, 0 < t < t_1$, obviously, $s(t) \in C^{3/4}[0, t_1]$, and for problem (AP), we have

LEMMA 1. *Under the assumptions* (A1) *and* (A2), *then problem* (AP) *has unique weak solution* $u \in C^{\lambda,\lambda/2}(\bar{Q}_{t_1}) \cap V_2^{1,0}(Q_{t_1})$ *classically satisfying* (3.5) *and* (3.6). *Moreover, the following equality and estimate hold:*

$$\int_0^1 u(x,t_1)\eta(x,t_1)\,dx - \int_0^{t_1}\int_0^1 u(x,t)\eta_t(x,t)\,dxdt$$

$$+ \int_0^{t_1}\int_0^1 [u_x(x,t) - g(b(t))H(s(t)-x)]\eta_x dxdt = \int_0^1 u_0(x)\eta(x,0)\,dx \qquad (3.8)$$

for all $\eta \in \overset{\circ}{W}{}_2^{1,1}(Q_{t_1})$

$$||u||_{C^{\lambda,\lambda/2}(\bar{Q}_{t_1})} \le K \ , \qquad (3.9)$$

where $\lambda \in (0,1)$ depends only on given data.

Proof. From [6], the existence and uniqueness of weak solutions are clear, so is (3.8). In order to get (3.9), we only need to set $r = q = 2$ in

[6:(7.2), p.181], then $k_1 = 1/4$ and choose $\bar{r} = \bar{q} = 4$, $\hat{r} = \hat{q} = 6$ and $k = 1/2$ in [6:(7.9)]. Owing to (3.3) we get (3.9). ■

Having got the solution $u(x,t)$ to problem (AP) we define

$$\sigma(t) = u(s(t),t), \quad 0 \leq t \leq t_1 . \tag{3.10}$$

Therefore, we have defined the operator F from E to $L^p[0,t_1]$ as follows:

$$F(b(t)) = \sigma(t) . \tag{3.11}$$

The operator F has the following properties:

LEMMA 2. *$FE \subset E$ and FE is precompact as t_1 sufficiently small.*

Proof. By means of (3.9), it is easy to get $FE \subset E$, in fact,

$$\int_0^{t_1} |\sigma(t)|^p dt = \int_0^{t_1} |u(s(t),t)|^p dt \leq K^p t_1 \leq K_1 .$$

To prove the compactness, we show that the FE is uniformly continuous, i.e. for any $\varepsilon > 0$, if $|t_0| < \delta$ sufficiently small, then

$$I := \int_0^{t_1} |\sigma(t_0 + t) - \sigma(t)|^p dt < \varepsilon . \tag{3.12}$$

Indeed,

$$\begin{aligned} I &= \int_0^{t_1} |u(s(t_0+t), t_0+t) - u(s(t),t)|^p dt \\ &\leq K \int_0^{t} \{|s(t_0+t) - s(t)|^\lambda + t_0^{\lambda/2}\}^p dt \\ &\leq K \int_0^{t_1} dt \left| \int_t^{t_0+t} |g(b(y))| dy \right|^{\lambda p} + K t_0^{p\lambda/2} t_1 \\ &\leq K \int_0^{t_1} dt \left| \int_t^{t_0+t} |b(y)|^m dy \right|^{\lambda p} + K t_0^{p\lambda/2} \leq K(t_0^{3\lambda p/4} + t_0^{p\lambda/2}) < \varepsilon \end{aligned}$$ ■

LEMMA 3. *Let b_1 and $b_2 \in E$, corresponding solutions to problem (AP) are u_1 and u_2 respectively and $\sigma_1 = F(b_1)$ and $\sigma_2 = F(b_2)$, then*

$$\|u_1(x,t) - u_2(x,t)\|_{L^\infty(Q_{t_1})} \leq K\delta^{1/8}, \text{ as } \|b_1 - b_2\|_{L^p\{0,t_1\}} < \delta .$$

Proof. By (3.8), let $u_{ih} = \int_t^{t+h} u_i(x,y)\,dy/h$ $(i = 1,2.)$, then u_{ih} satisfies the following equality (see [6:(2.12), p.142]):

$$\int_{Q_{t_2}} (u_{ih})_t\eta + [u_{ix} + g(b_i)H(s_i - x)]_h\eta_x = 0 \text{ for all } \eta \in \overset{\circ}{V}{}_2^{1,0}(Q_{t_2})$$

$$0 \leq t_2 < t_1 - h . \tag{3.13}$$

Setting $w = u_1 - u_2$, then $w \in \overset{\circ}{V}{}_2^{1,0}(Q_{t_2})$, from (3.13), we get

$$\int_{Q_{t_2}} (w_{ht})\eta + [w_x + g(b_1)H(s_1 - x) - g(b_2)H(s_2 - x)]_h\eta_x = 0 . \tag{3.14}$$

Choosing $\eta = w_h^{(k)} = \max\{w_h - k, 0\}, k \geq 0$, in (3.14), let $h \to 0$ we have

$$\int_0^1 w^{(k)}(x,t_1)^2 dx + \int_0^{t_1}\int_0^1 w_x^{(k)}(x,t)^2 dxdt$$

$$\leq K \int_{Q_{t_1}\cap\{w>k\}} [g(b_1)H(s_1 - x) - g(b_2)H(s_2 - x)]^2\, dxdt . \tag{3.15}$$

Denoting the right integration of (3.15) by J and $\text{mes}\{x : w(x,t) > k\} = A_k(t)$, $\min(s_1, s_2) = s_*$ and $\max(s_1, s_2) = s^*$, we obtain

$$\begin{aligned} J = &\int_0^{t_1}\int_0^{S_*\cap\{w>k\}} (g(b_1) - g(b_2))^2 dxdt \\ &+ \int_0^{t_1}\int_{S_*\cap\{w>k\}}^{S^*\cap\{w>k\}} [g(b_1)H(s_1 - x) - g(b_2)H(s_2 - x)]^2 dx \\ \leq &\int_0^{t_1} dt \int_0^1 |g'(y)|^2 |b_1 - b_2|^2 dx \end{aligned}$$

$$+ K\int_0^{t_1} dt \int_{S_*\cap\{w>k\}}^{S^*\cap\{w>k\}} [|b_1|^{2m} + |b_2|^{2m} + 1]dx$$

$$(y = b_1 + \theta(b_2 - b_1),\ 0 < \theta < 1)$$

$$\le K\int_0^{t_1} A_k(t)(|b_1|^{2m-2} + |b_2|^{2m-2} + 1)(b_1 - b_2)^2 dt$$

$$+ K\int_0^{t_1} (|b_1|^{2m} + |b_2|^{2m} + 1)|s_1 - s_2|^{1/2} A_k^{1/2}(t)dt$$

$$\le K\left(\int_0^{t_1} |b_1 - b_2|^p\right)^{2/p} \left(\int_0^{t_1} |b_1|^p + |b_2|^p + 1\right)^{\frac{m-1}{2m}} \left(\int_0^{t_1} A_k^2(t)\right)^{1/2}$$

$$+ K\max_{0\le t\le t_1} |s_1 - s_2|^{1/2} \left(\int_0^{t_1} |b_1| + |b_2|^p + 1\right)^{1/2} \left(\int_0^{t_1} A_k(t)\,dt\right)^{1/2}$$

$$\le K\delta^2 \left(\int_0^{t_1} A_k^2(t)\right)^{1/2} + K\delta^{1/2}\left(\int_0^{t_1} A_k(t)\right)^{1/2}. \tag{3.16}$$

Here we have used the following estimate:

$$|s_1(t) - s_2(t)| \le \int_0^t |g(b_1) - g(b_2)|dt$$

$$\le K\int_0^t (|b_1|^{m-1} + |b_2|^{m-1} + 1)|b_1 - b_2|$$

$$\le K\left(\int_0^t |b_1 - b_2|^p\right)^{1/p} \left(\int_0^t (|b_1|^{m-1} + |b_2|^{m-1} + 1)^{p/(p-1)}\right)^{(p-1)} \le K\delta. \tag{3.17}$$

From (3.15) and (3.16) we get

$$|w^{(k)}|_{Q_{t_1}}^2 \le K\delta^{1/2}\left(\int_0^{t_1} A_k(t)\,dt\right)^{1/2}. \tag{3.18}$$

Writing $\mu(k) := \int_0^t A_k(t)\,dt$ and setting $k_h = \delta^{1/8}(2 - 2^{-k})$, $h = 0, 1, 2, \ldots$, we obtain

$$(k_{h+1} - k_h)\mu^{1/6}(k_{h+1}) \le \left(\int_0^{t_1}\int_0^1 (w^{(k_h)})^6\right)^{1/6}.$$

On the other hand, by the imbedding theorem in [6: p, 75], we get

$$\left[\int_0^{t_1}\int_0^1 [w^{(k_h)}]^6\right]^{1/6} \le K\beta|w^{(k_h)}|_{Q_{t_1}} \le K\delta^{1/4}\beta\mu^{1/4}(k_h) .$$

So,

$$\mu^{1/6}(k_{h+1}) \le K\beta\delta^{1/4}\mu(k_h)^{1/6(1+k)}/(k_{h+1}-k_h)$$
$$= k\beta\delta^{1/8}2^h\mu(k_h)^{1/6(1+k)}(k=1/2) . \qquad (3.19)$$

Hence we obtain $\mu^{1/6}(k_h) \to 0$ as $h \to \infty$ provided the following condition holds (see [6:Lemma 5.6]):

$$\mu^{1/6}(k_0) = \mu^{1/6}(\delta^{1/8}) \le 2^{-4}(K\beta\delta^{1/8})^{-2} . \qquad (3.20)$$

This is clear if δ is sufficiently small. So, $\mu^{1/6}(2\delta^{1/8}) = 0$, i.e., $w \le 2\delta^{1/8}$. similarly, we get the estimate of $u_2 - u_1$. ∎

LEMMA 4. *The operator F is continuous.*

Proof. Let b_i, σ_i and $u_i (i = 1, 2.)$ be the functions in Lemma 3 and $||b_1 - b_2||_{L^p[0,t_1]} \le \delta$. For any $\varepsilon > 0$, we need to show, as δ small

$$I := ||\sigma_1 - \sigma_2||_{L^p[0,t_1]} < \varepsilon . \qquad (3.21)$$

First, we choose δ so small, such that Lemma 3 holds. Then

$$I = \left(\int_0^{t_1} |\sigma_1(t) - \sigma_2(t)|^p dt\right)^{1/p} = \left(\int_0^{t_1} |u_1(s_1)(t), t) - u_2(s_2(t), t||^P\right)^{1/p}$$
$$\le \left[\int_0^{t_1} (u_1(s_1(t), t) - u_1(s_2(t), t)|^p dt\right]^{1/P}$$
$$+ \left[\int_0^{t_1} |u_1(s_2(t), t) - u_2(s_2(t), t)|^P dt\right]^{1/2} . \qquad (3.22)$$

The second integration of (3.22) is less than ε because of Lemma 3 and so is the first as Lemma 1 and (3.17). ∎

From Lemma 2 and 4, using Schauder's fixed point theorem, we finish the proof of Theorem 1.

In order to prove Theorem 1', from Theorem 1, we first get the solution

$$u(x,t) \in C^{\lambda,\lambda/2}(\bar{Q}_{t_1}) \text{ and } s(t) \in C^{1+\lambda/2}[0,t_1] .$$

Next, we consider the auxiliary problem (BP) as follows:

$$(\mathrm{BP}) \begin{cases} v_t - v_{xx} = 0 , & \text{in } Q_{1t_1} \cup Q_{2t_1} , \\ v(0,t) = h_1(t),\ v(1,t) = h_2(t) , & 0 < t < t_1 , \\ v(x,0) = u_0(x) , & 0 < x < 1 , \\ v(s(t)+,t) = v(s(t)-,t), & \\ v_x(s(t)t+,t) - v_x(s(t)-,t) = g(u(s(t),t)) , & 0 < t < t_1 . \end{cases}$$

By [5: Theorem 1], the problem (BP) has a classical solution $v(x,t)$. Thus (3.13) with $b_i = s$ and $u_i = v$ or u holds. Setting $w = v - u$ and $\eta = w_h$, we obtain

$$\int_{Q_{t_2}} [(-w_h)(w_h)_t + w_{hx}^2] dxdt = 0 . \tag{3.23}$$

Let $h \to 0$, we have

$$\int_0^1 w^2(x,t_2)dx + \int_{Q_{t_2}} w_x^2(x,t)dxdt = 0 .$$

Hence $w = 0$, i.e. $v = u$. Therefore, we have proved Theorem 1'.

Proof of Theorem 2 *and* 2'. From Theorem 1, we can extend the solution $\{u(x,t),s(t)\}$ to the more large domain Q_{t_2}, with $t_2 \in (t_1,T]$, then Q_{t_3} with $t_3 \in (t_4,T]$, and so on. Therefore, we obtain the increasing sequence $t_1,t_2,t_3,\ldots$ and the corresponding sequence of solutions $\{u_1(x,t),s_1(t)\}$ in $Q_{t_{i-1},t_i} = \bar{Q}_{t_1} \backslash Q_{t_{i-1}}$ with $t_0 = 0$, $u_1 = u$ and $s_1 = s$ $(i = 1,2,3\ldots)$ obviously setting $u(x,t) = u_i(x,t)$ and $s(t) = s_i(t)$ in $Q_{t_{i-1}t_i}$ $(i = 1,2,3,\ldots,)$, we get the weak solution $\{u(x,t),s(t)\}$ in Q_{t_ℓ} for any positive integer ℓ. The difference $t_i - t_{i-1}$, however, depends only on $u(x,t_{i-1})$, $s(t_{i-1})$ and $1 - s(t_{i-1})$ besides the given data because of the above proofs of Theorem 1.

Thus three cases may be deduced as $i \to \infty$:

CASE 1: $t_i \to T$ and $0 < s(t) < 1, 0 < t < T$;
CASE 2: $t_i \to T_1, T_1 < T$ but $s(T_1) = 0$ or $s(T_1) = 1$;
CASE 3: $t_i \to T_1, T_1 < T$ but $0 < s(t) < 1, 0 \le t \le T_1$.

For Case 3, we have

LEMMA 5. *The Case* 3 *does not occur.*

Proof. It is enough to show that the L^∞ norm of $u(x,t)$ depends only on the given data. Indeed, this is true as $|g(y)| \le C(|y| + 1)$ (see [4: Proposition 2]). When $yg(y) \ge 0$, let M be the L_∞ bound of $h_1(t), h_2(t)$ and $u_0(x)$, choosing

$$\eta(x,t) = \int_t^{t+h} \max(u(x,y) - M, 0)\,dy/h = \int_t^{t+h} (u(x,y) - M)^+ dy/h$$

in (3.8) for any $t_1 < T_1$ and $b(t) = u(s(t),t)$ and let $h \to 0$, we get

$$\int_0^1 u(x,t_1)(u(x,t_1) - M)^+ dx + \int_{Q_{t_1}} [(u(x,t) - M)_x^+]^2 dx dt$$

$$+ \int_0^{t_1} g(u(s(t),t))(u(s(t),t) - M)^+ dt = 0 .$$

Thus, $u < M$ and so does $-u < M$ in Q_{t_1}. ■

From Lemma 5, we complete the proof of Theorem 2. The proof of Theorem 2′ is the same as that of Theorem 1′.

4. Some remarks.

(1) We can obtain the same results for a more general equation:

$$a(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial u}{\partial x} + c(x,t)u - \frac{\partial u}{\partial t} = 0$$

than (1.1) under some conditions for a, b and c (see [5]).

(2) As the condition (A1)′ is not true, for instance, $g(y) = y^{2m}$, where m is positive integer, the solution to problem (NP) exists either in

the whole domain Q_T or in a proper subdomain $Q_{T_1} \subset Q_T$ with $\lim \max |u(x,t)| = \infty$ as $0 < s(t) < 1,\ 0 \leq t \leq T_1$. But we do not know which case occurs

(3) The problem of uniqueness is still open.

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