SOME CONSEQUENCES OF AN EASY CARDINAL INEQUALITY INVOLVING SEPARATING OPEN COVERS (*)

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SOMMARIO. - *Vengono fornite alcune disuguaglianze cardinali relative a varie funzioni cardinali definite in termini di certi ricoprimenti aperti. Tra l'altro si prova che* $|X| \leq e(X)^{\psi_m(X)}$ *e* $|X| \leq wL(X)^{\psi_u(X)}$ *per ogni* T_1*-spazio* X *completamente regolare. Qui* $e(X)$, $wL(X)$, $\psi_m(X)$ *e* $\psi_u(X)$ *denotano rispettivamente l'estensione, il numero debole di Lindelöf, lo pseudo grado di metrizzabilità e lo pseudo peso uniforme di* X.

SUMMARY. - *Some cardinal inequalities with cardinal functions defined in terms of certain types of covers are given. Among other results it is shown that* $|X| \leq e(X)^{\psi_m(X)}$ *and* $|X| \leq wL(X)^{\psi_u(X)}$ *for any completely regular* T_1*-space* X*. Here* $e(X)$, $wL(X)$, $\psi_m(X)$, $\psi_u(X)$ *denote respectively the extent, the weak Lindelöf number, the pseudo-metrizability degree and the pseudo uniform weight of* X.

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The aim of this note is to present some cardinal inequalities proved with the help of an estimate of the cardinality of a T_1-space in terms of a certain separating open covers.

Two of the results stated here are improvements of inequalities given in [4].

Our notation concerning cardinal functions follows [6]. $|S|$ denotes the cardinality of the set S.

Given a family $\mathcal{V} \cup (A)$ of subsets of S we denote by $\mathcal{V}[A]$, $\mathrm{ord}(A, \mathcal{V})$ and $\mathrm{st}(A, \mathcal{V})$ respectively $\{V : V \in \mathcal{V} \text{ and } V \cap A \neq \emptyset\}$, $|\{V : V \in \mathcal{V} \text{ and } V \cap A \neq \emptyset\}|$ and $\cup\{V : V \in \mathcal{V} \text{ and } V \cap A \neq \emptyset\}$. When $A = \{x\}$ we

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simply write st$(x, \mathcal{V})$ and ord$(x, \mathcal{V})$.

$\mathrm{st}^{i+1}(x, \mathcal{V})$ is inductively defined by $\mathrm{st}^{i+1}(x, \mathcal{V}) = \mathrm{st}(\mathrm{st}^{i}(x, \mathcal{V}), \mathcal{V})$. For completeness we recall the definitions of the cardinal functions used throughout the paper.

$L(X)$, the Lindelöf number of X, is the smallest cardinal number k such that any open cover of X has a subcover of cardinality not exceeding k. $wL(X)$, the weak Lindelöf number of X, is the smallest cardinal number k such that any open cover of X has a weak subcover of cardinality not exceeding k. A collection is said to be a weak cover provided that its union is dense in X.

$d(X)$, the density of X, is the smallest cardinality of a dense subset of X. $e(X)$, the extent of X, is defined as $\sup\{|A| : A$ closed discrete subset of $X\}$.

$\psi^{i}_{m}(X)$ (see [4]) is the smallest cardinal number k such that there exists a collection $\{\mathcal{G}_\alpha : \alpha \in k\}$ of open covers of X with $\bigcap_{\alpha\in k} \mathrm{st}^{i+1}(X, \mathcal{G}_\alpha) = \{x\}$ for any $x \in X$. The definability of the previous function for $i \geq 1$ is certainly guaranteed in the class of regular T_1-spaces.

$\psi^{0}_{m}(X)$ is the well-known diagonal degree $\psi_\Delta(X)$ and $\psi^{1}_{m}(X)$ is the pseudo metrizability degree $\psi_m(X)$ introduced in [3].

$\psi_u(X)$, the pseudo uniform weight of X (see [2]), is the smallest cardinal number k such that the diagonal of X is the intersection of k entourages of a compatible uniformity of X.

psw(X), the point separating weight of X, is the smallest cardinal number k such that there exists a separating open cover $\mathcal{V}$ of X such that ord$(x, \mathcal{V}) \leq k$ for any $x \in X$; $\mathcal{V}$ is said to be separating provided that for any $x \neq y$ there exists some $V \in \mathcal{V}$ such that $x \in V$ and $y \notin V$.

ssw(X), the star separating weight of X, is the smallest cardinal number k such that there exists a separating open cover $\mathcal{V}$ of X such that ord$(V, \mathcal{V}) \leq k$ for any $V \in \mathcal{V}$.

To establish the fundamental lemma we need the following:

DEFINITION. Let X be a T_1-space and $\mathcal{V}$ a separating open cover of X. $\eta(\mathcal{V})$ is the smallest cardinal number k such that there exists a map $f : X \times k \to \mathcal{V}$ satisfying $\bigcap_{\alpha\in k} f(x, \alpha) = \{x\}$ for any $x \in X$.

Although the cardinal number η plays an important role in the sequel, it should be noticed that the function defined as the minimum of η does not bring any new concept in the theory.

Indeed we have $\min\{\eta(\mathcal{V}):\ \mathcal{V}$ a separating open cover of $X\} = \psi(X)$, where $\psi(X)$ is the pseudo character of X.

LEMMA. *If $\mathcal{V}$ is a separating open cover of X then* $|X| \leq |\mathcal{V}|^{\eta(\mathcal{V})}$.

Proof. Let $f : X \times \eta(\mathcal{V}) \to \mathcal{V}$ be the defining function of η.

Since $\bigcap_{\alpha\in\eta(\mathcal{V})} f(x,\alpha) = \{x\}$ for any $x \in X$, it turns out that the correspondence $x \to \{f(x_\alpha) : \alpha \in \eta(\mathcal{V})\}$ is injective and consequently $|X| \leq |\mathcal{V}|^{\eta(\mathcal{V})}$.

The next two inequalities were obtained by O.T. Alas in [1].

PROPOSITION 1. *If X is a T_1-space then* $|X| \leq L(X)^{\psi_\Delta(X)}$.

Proof. Let $\{\mathcal{G}_\alpha : \alpha \in \psi_\Delta(X)\}$ be a collection of open covers of X such that $\bigcap_{\alpha\in\psi_\Delta(X)} \mathrm{st}(x,\mathcal{G}_\alpha) = \{x\}$ for any $x \in X$. For any $\alpha \in \psi_\Delta(X)$ choose a subcover $\mathcal{V}_\alpha$ of $\mathcal{G}_\alpha$ with cardinality not exceeding $L(X)$ and let $\mathcal{V} = \bigcup_{\alpha\in\psi_\Delta(X)} \mathcal{V}_\alpha$.

Clearly for such a cover we have $\eta(\mathcal{V}) \leq \psi_\Delta(X)$ and by the Lemma we get $|X| \leq |\mathcal{V}|^{\eta(\mathcal{V})} \leq (L(X)\psi_\Delta(X))^{\psi_\Delta(X)} = L(X)^{\psi_\Delta(X)}$.

PROPOSITION 2. *If X is a T_1-space then* $|X| \leq d(X)^{\mathrm{psw}(X)}$.

Proof. Let $\mathcal{V}$ be a separating open cover of X such that $\mathrm{ord}(x,\mathcal{V}) \leq \mathrm{psw}(X)$ for any $x \in X$.

If A is a dense subset of X then $|\mathcal{V}| \leq |A|\mathrm{psw}(X)$ and consequently choosing A such that $|A| = d(X)$, we have $|X| \leq |\mathcal{V}|^{\eta(\mathcal{V})} \leq d(X)^{\mathrm{psw}(X)}$.

PROPOSITION 3. *If X is a regular T_1-space then* $|X| \leq e(X)^{\psi_m(X)}$.

Proof. Let $\{\mathcal{G}_\alpha\colon \alpha \in \psi_m(X)\}$ be a collection of open covers such that $\bigcap_{\alpha\in\psi_m(X)} \mathrm{st}^2(x,\mathcal{G}_\alpha) = \{x\}$ for any $x \in X$. For any $\alpha \in \psi_m(X)$ let D_α be a subset of X maximal with respect to the property that $\mathrm{st}(x,\mathcal{G}_\alpha)\cap D_\alpha=\{x\}$ for any $x\in D_\alpha$.

The set D_α exists by Zorn's lemma and it is easily seen it is closed and discrete. The maximality of D_α implies that the family $\{\mathrm{st}(x,\mathcal{G}_\alpha):x\in D_\alpha\}$ is a cover of X of cardinality not exceeding $e(X)$.

We claim that the family $\mathcal{V}=\{\mathrm{st}(x,\mathcal{G}_\alpha): x\in D_\alpha, \alpha\in\psi_m(X)\}$ is separating. Indeed let $p,q\in X$ and $p\neq q$.

By definition of pseudometrizability degree there exists some $\alpha\in\psi_m(X)$ such that $q\notin \mathrm{st}^2(p,\mathcal{G}_\alpha)$. For such α let $x\in D_\alpha$ so that $p\in\mathrm{st}(x,\mathcal{G}_\alpha)$ and observe that q cannot belong to $\mathrm{st}(x,\mathcal{G}_\alpha)$ since otherwise it would be $q\in\mathrm{st}^2(p,\mathcal{G}_\alpha)$.

We have $\eta(\mathcal{V})\leq\psi_m(X)$ and therefore $|X|\leq|\mathcal{V}|^{\eta(\mathcal{V})}\leq e(X)^{\psi_m(X)}$.

Comparing the inequalities in propositions 1 and 3 with the Ginsburg-Wood's formula $|X|\leq 2^{e(X)\psi_\Delta(X)}$ for any T_1-space X (see [5]), it is natural to ask the following question "is it true that $|X|\leq e(X)^{\psi_\Delta(X)}$ for any regular T_1-space X"?

PROPOSITION 4. *If X is a T_1-space then $|X|\leq wL(X)^{\mathrm{ssw}(X)}$.*

Proof. Let $\mathcal{V}$ be a separating open cover of X such that $\mathrm{ord}(V,\mathcal{V})\leq\mathrm{ssw}(X)$ for any $V\in\mathcal{V}$. If $\mathcal{G}$ is a weak cover of $\mathcal{V}$ then we have $\mathcal{V}=\bigcup_{G\in\mathcal{G}}\mathcal{V}[G]$ and consequently $|\mathcal{V}|\leq|\mathcal{G}|\mathrm{ssw}(X)$.

Since $\eta(\mathcal{V})\leq\mathrm{ssw}(X)$ we obtain, choosing $|\mathcal{G}|=wL(X)$, $|X|\leq|\mathcal{V}|^{\eta(\mathcal{V})}\leq wL(X)^{\mathrm{ssw}(X)}$.

PROPOSITION 5. *If X is a regular T_1-space then $|X|\leq wL(X)^{\psi_m^2(X)}$.*

Proof. Let $\{\mathcal{G}_\alpha:\alpha\in\psi_m^2(X)\}$ be a collection of open covers of X such that $\bigcap_{\alpha\in\psi_m^2(X)}\mathrm{st}^3(x,\mathcal{G}_\alpha)=\{x\}$ for any $x\in X$.

For any $\alpha\in\psi_m^2(X)$ choose a weak subcover $\mathcal{X}_\alpha$ of $\mathcal{G}_\alpha$ with $|\mathcal{X}_\alpha|\leq wL(X)$ and let $\mathcal{V}_\alpha=\{\mathrm{st}(H,\mathcal{G}_\alpha):H\in\mathcal{X}_\alpha\}$. $\mathcal{V}=\bigcup_{\alpha\in\psi_m^2(X)}\mathcal{V}_\alpha$ is separating. Indeed let $p\neq q$ and choose $\alpha\in\psi_m^2(X)$ such that $q\notin\mathrm{st}^3(p,\mathcal{G}_\alpha)$.

Let $H\in\mathcal{X}_\alpha$ so that $p\in\mathrm{st}(H,\mathcal{G}_\alpha)$ and notice that $q\notin\mathrm{st}(H,\mathcal{G}_\alpha)$ since otherwise it would be $q\in\mathrm{st}^3(p,\mathcal{G}_\alpha)$. $\eta(\mathcal{V})\leq\psi_m^2(X)$ and $|\mathcal{V}|\leq wL(X)\psi_m^2(X)$ and therefore $|X|\leq wL(X)^{\psi_m^2(X)}$. The inequalities in propositions 4 and 5 improve the inequalities $|X| \leq 2^{wL(X)\mathrm{ssw}(X)}$ and $|X| \leq 2^{wL(X)\psi_m^2(X)}$ given in [4].

In [4] it is proved that $\psi_m^2(X) \leq \psi_u(X)$ for any completely regular T_1-space X thus we have:

COROLLARY. If X is a completely regular T_1-space then $|X| \leq wL(X)^{\psi_u(X)}$.

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