# EIGENVALUES AND EIGENVECTORS OF A SPECIAL CLASS OF BAND MATRICES (*)

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SOMMARIO. -*Scopo di questo lavoro è fornire autovalori ed autovettori esatti di una classe di particolari matrici, che possono essere usate come matrici test. Sono state considerate a tal fine dapprima matrici di Toeplitz tridiagonali, quindi matrici ottenute da queste con opportune correzioni ed infine alcune matrici di Haenkel, legate alle precedenti da una relazione di similitudine.*

SUMMARY. -*The aim of this paper is to give exact eigenvalues and eigenvectors of a class of special matrices, to be used in testing algorithms. To this end, first Toeplitz tridiagonal matrices are considered, then matrices obtained from them by corrections and after a class of Haenkel matrices related to the privious by similarity transformations.*

**Introduction.**

In order to find eigenvalues and eigenvectors, the following considerations have been taken into account.

i) Let $A$ be a matrix of order $n$. We can associate to the equation $Ax = \lambda x$ a $(n - 1)$ -order difference equation, with suitable initial conditions. In the special case of a Toeplitz matrix, the coefficients of such equation are constant and if, in addition, the matrix is tridiagonal, its order is two. If we take into account the relations between roots and coefficients of the characteristic equation associated to this difference equation, then the general solution (with initial conditions) in some cases allows to find eigenvalues and eigenvectors.

ii) Let $\lambda$ be an eigenvalue of $A$, $\alpha$, $t$ scalars, $I$ the identity matrix; then $\alpha\lambda + t$ is an eigenvalue of $B = \alpha A + tI$; eigenvectors of $A$ and $B$, associated to $\lambda$ and $\alpha\lambda + t$ respectively, are the same.

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iii) Let $A = S^{-1}BS$, $y$ a right eigenvector of $B$ associated to the eigenvalue $\lambda$; then $x = S^{-1}y$ is an eigenvector of $A$ associated to $\lambda$. If, as a special case, $S$ is a permutation matrix, the elements of $x$ are a permutation of those of $y$.

iv) The roots of the algebraic equation $z^n = 1$ are

$$z_s = e^{i\frac{2s}{n}\pi} = \cos\frac{2s}{n}\pi + i\sin\frac{2s}{n}\pi, \quad s = 0,\dots,n-1,$$

those of $z^n = -1$ are

$$z_s = e^{i\frac{2s-1}{n}\pi} = \cos\frac{2s-1}{n}\pi + i\sin\frac{2s-1}{n}\pi, \quad s = 0,\dots,n-1.$$

In what follows, $e_s$ is the $s$-th element of canonical base in $R^n$, $O_{nxm}$ the null matrix sized $nxm$, $J_n = [e_n, e_{n-1}, \dots, e_1]$; we shall write simply $O$ or $J$ if there is no ambiguity. If $A = JBJ$ or $A = kB$ ($k$ scalar), $A$ and $B$ will not be regarded as different ones.

**1. Tridiagonal Toeplitz matrices.**

1.1 The $n$-order matrix

$$T_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

has the following eigenvalues and eigenvectors, respectively,

$$\lambda_s = 2\cos\frac{\pi}{n+1}s, \quad s = 1,\dots,n;$$

$$x_j^{(s)} = \sqrt{\frac{2}{n+1}}\ \sin \frac{\pi}{n+1} js \ , \ j = 1,\dots,n \ ; \ s = 1,\dots n.$$

Eigenvalues and eigenvectors of

$$T_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix}$$

are respectively:

if $n = 2k + 1$

$$\lambda_s = 2i \sin \frac{s}{2k+2} \pi \ , \ s = -k,\dots,k \ ;$$

$$x_j^{(s)} = \left[1+(-1)^{j+1}\right] \cos \frac{sj}{2k+2} \pi + i \left[1+(-1)^{j+1}\right] \sin \frac{sj}{2k+2} \pi$$

$$j = 1,\dots,n \quad ; \quad s = -k,\dots,k \ ;$$

if $n = 2k$

$$\lambda_s = 2i \sin \frac{2s-1}{4k+2} \pi \ , \ s = -k+1,\dots,k \ ;$$

$$x_j^{(s)} = \left[1+(-1)^{j+1}\right] \cos \frac{j(2s-1)}{4k+2} \pi + i \left[1+(-1)^{j+1}\right] \sin \frac{j(2s-1)}{4k+2} \pi \ ,$$

$$j = 1,\dots,n \quad ; \quad s = -k+1,\dots,k \ .$$

1.2. LEMMA. *Let $T_1$, $T_2$ be defined as in 1.1, $b, c \neq 0$, $\alpha = sg(b)\sqrt{|bc|}$,*

$$T = \begin{bmatrix} a & b & 0 & \cdots & 0 & 0 \\ c & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & c & a \end{bmatrix}.$$

*Then $T$ is similar either to $\alpha(T_1 + \frac{a}{\alpha}I)$ or to $\alpha(T_2 + \frac{a}{\alpha}I)$ depending if $bc > 0$ or $bc < 0$, respectively.*

*Proof.* Let $D = \mathrm{diag}\,\{d_1, d_2, \ldots d_n\}$, where $d_1 = \sqrt{|bc|}$, $d_i = d_{i-1}\sqrt{|c/b|}$, $i = 2, \ldots, n$. Then

$$\hat{T} = D^{-1}TD = \begin{bmatrix} a & sg(b)d_1 & 0 & \cdots & 0 & 0 \\ sg(c)d_1 & a & sg(b)d_1 & \cdots & 0 & 0 \\ 0 & sg(c)d_1 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & sg(b)d_1 \\ 0 & 0 & 0 & \cdots & sg(c)d_1 & a \end{bmatrix}$$

is similar to $T$. If we set $\alpha = sg(b)d_1$, $\hat{T} = \alpha\left(T_1 + \frac{a}{\alpha}I\right)$ if $bc>0$, $\hat{T} = \alpha\left(T_2 + \frac{a}{\alpha}I\right)$ if $bc<0$, Q.E.D..

1.3. THEOREM. *Let T, D, α be defined as in* 1.2. *Then eigenvalues and eigenvectors of T are, respectively*:

*if* $bc > 0$

$$\lambda_s = a + 2\alpha\cos\frac{s}{n+1}\pi \ , \ s = 1,\dots,n \ ;$$

$$x_j^{(s)} = d_j \sin\frac{js}{n+1}\pi \ , \ j = 1,\dots,n \ ; \ s = 1,\dots,n \ ;$$

*if* $bc < 0$

*for* $n = 2k+1$

$$\lambda_s = a + 2\alpha i \sin\frac{s}{2k+2}\pi \ , \ s = -k,\dots,k \ ;$$

$$x_j^{(s)} = d_j\left\{\left[1+(-1)^{j+1}\right]\cos\frac{js}{2k+2}\pi + i\left[1+(-1)^{j+1}\right]\sin\frac{js}{2k+2}\pi\right\} ,$$

$$j = 1,\dots,n \ ; \ s = -k,\dots,k \ ;$$

*for* $n = 2k$

$$\lambda_s = a + 2\alpha i \sin\frac{2s-1}{4k+2}\pi \ , \ s = -k+1,\dots,k \ ;$$

$$x_j^{(s)} = d_j\left\{\left[1+(-1)^{j+1}\right]\cos\frac{j(2s-1)}{4k+2}\pi + i\left[1+(-1)^{j+1}\right]\sin\frac{j(2s-1)}{4k+2}\pi\right\} ,$$

$$j = 1,\dots,n \ ; \ s = -k+1,\dots,k \ .$$

*Proof.* It follows immediately from 1.2, taking into account the considerations of introduction, Q.E.D..

1.4. COROLLARY. *The matrix*

$$T = \begin{bmatrix} a & b & 0 & \cdots & 0 & 0 \\ b & a & b & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & b & a \end{bmatrix}$$

*has eigenvalues and eigenvectors, respectively,*

$$\lambda_s = a + 2b \cos \frac{s}{n+1} \pi \ , \ s = 1,...,n \ ;$$

$$x_j^{(s)} = \sqrt{\frac{2}{n+1}} \ \sin \frac{js}{n+1} \pi \ , \ j = 1,...,n \ ; \ s = 1,...n.$$

**2. Matrices obtained from Toeplitz tridiagonal ones.**

We shall give now eigenvalues and eigenvectors of a collection of tridiagonal matrices, obtained from the Toeplitz tridiagonal simmetric matrix defined in 1.4. by changing at most two elements.

2.1. Let

$$T_3 = \begin{bmatrix} a-b & b & 0 & \cdots & 0 & 0 \\ b & a & b & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & b & a \end{bmatrix};$$

it has eigenvalues and eigenvectors, respectively:

$$\lambda_s = a + 2b \cos\frac{2s}{2n+1}\pi \ , \ s = 1,\dots,n \ ;$$

$$x_j^{(s)} = \frac{2}{\sqrt{2n+1}} \ \sin \frac{(2j-1)s}{2n+1}\pi \ , \ j = 1,\dots,n \ ; \ s = 1,\dots n.$$

2.2. Let

$$T_4 = \begin{bmatrix} a+b & b & 0 & \cdots & 0 & 0 \\ b & a & b & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & b & a \end{bmatrix};$$

its eigenvalues and eigenvectors are, respectively:

$$\lambda_s = a + 2b \cos\frac{2s-1}{2n+1}\pi \ , \ s = 1,\dots,n \ ;$$

$$x_j{}^{(s)} = \frac{2}{\sqrt{2n+1}} \ \sin \frac{(n-j+1)\,(2s-1)}{2n+1}\pi \ , \ j = 1,...,n \ ; \ s = 1,...n.$$

2.3. The matrix

$$T_5 = \begin{bmatrix} a & 2b & 0 & \cdots & 0 & 0 \\ b & a & b & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & b & a \end{bmatrix}$$

has eigenvalues and eigenvectors, respectively:

$$\lambda_s = a + 2b\cos\frac{2s-1}{2n}\pi \ , \ s = 1,...,n \ ;$$

$$x_j{}^{(s)} = \ \sin \frac{(n-j+1)\,(2s-1)}{2n}\pi \ , \ j = 1,...,n \ ; \ s = 1,...n.$$

2.4. Let

$$T_6 = \begin{bmatrix} a+b & b & 0 & \cdots & 0 & 0 \\ b & a & b & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & b & a+b \end{bmatrix};$$

its eigenvalues and eigenvectors are, respectively:

$$\lambda_s = a + 2b \cos \frac{s-1}{n} \pi \ , \ s = 1,\dots,n \ ;$$

$$x_j{}^{(1)} = \sqrt{\frac{1}{n}} \ , \ j = 1,\dots,n \ ;$$

$$x_j{}^{(s)} = \sqrt{\frac{2}{n}} \cos \frac{(2j-1)(s-1)}{2n} \pi \ , \ j = 1,\dots,n \ ; \ s = 2,\dots,n \ .$$

2.5. The matrix

$$T_7 = \begin{bmatrix} a-b & b & 0 & \cdots & 0 & 0 \\ b & a & b & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & b & a+b \end{bmatrix}$$

has eigenvalues and eigenvectors, respectively:

$$\lambda_s = a + 2b \cos \frac{2s-1}{2n} \pi \ , \ s = 1,\dots,n \ ;$$

$$x_j{}^{(s)} = \sqrt{\frac{2}{n}} \sin \frac{(2j-1)(2s-1)}{4n} \pi \ , \ j = 1,\dots,n \ ; \ s = 1,\dots,n \ .$$

2.6. Let

$$T_8 = \begin{bmatrix} a-b & b & 0 & \cdots & 0 & 0 \\ b & a & b & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & b & a-b \end{bmatrix};$$

it has eigenvalues and eigenvectors, respectively:

$$\lambda_s = a + 2b \cos \frac{s}{n} \pi \ , \ s = 1,\ldots,n \ ;$$

$$x_j^{(s)} = \sqrt{\frac{2}{n}} \sin \frac{(2j-1)\,s}{2n} \pi \, , \ j = 1,\ldots,n \ ; \ s = 1,\ldots,n \, .$$

2.7. The matrix

$$T_9 = \begin{bmatrix} a & 2b & 0 & \cdots & 0 & 0 \\ b & a & b & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & 2b & a \end{bmatrix}$$

has eigenvalues and eigenvectors, respectively:

$$\lambda_s = a + 2b \cos \frac{s-1}{n-1} \pi \ , \ s = 1,\ldots,n \ ;$$

$$x_j^{(s)} = \cos \frac{(j-1)(s-1)}{n-1}\pi \, , \; j = 1,\ldots,n \; ; \; s = 1,\ldots,n \, .$$

2.8. Let

$$T_{10} = \begin{bmatrix} a+b & b & 0 & \cdots & 0 & 0 \\ b & a & b & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & 2b & a \end{bmatrix};$$

its eigenvalues and eigenvectors are, respectively:

$$\lambda_s = a + 2b \cos \frac{2(s-1)}{2n-1}\pi \;\; , \;\; s = 1,\ldots,n \;\; ;$$

$$x_j^{(s)} = \cos \frac{(2j-1)(s-1)}{2n-1}\pi \, , \; j = 1,\ldots,n \; ; \; s = 1,\ldots,n \, .$$

2.9 The matrix

$$T_{11} = \begin{bmatrix} a-b & b & 0 & \cdots & 0 & 0 \\ b & a & b & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & 2b & a \end{bmatrix}$$

has eigenvalues and eigenvectors, respectively:

$$\lambda_s = a + 2b\cos\frac{2s-1}{2n-1}\pi \ , \ s = 1,\dots,n \ ;$$

$$x_j^{(s)} = \sin\frac{(2j-1)\,(2s-1)}{2(2n-1)}\pi \, , \, j = 1,\dots,n \ ; \ s = 1,\dots,n \, .$$

## 3. A special Haenkel matrix and matrices obtained from it.

We shall deal now with a special Haenkel matrix and a class of matrices obtained from it by correcting at most two elements. In what follows, we set

$$P = \begin{bmatrix} I & I \\ J & -J \end{bmatrix} \quad , \quad \widetilde{P} = \begin{bmatrix} I & O & I \\ O & 1 & O \\ J & O & -J \end{bmatrix}$$

(whose orders are $n = 2k$ and $n = 2k+1$), where $I$ and $J$ have order $k$, while $O$ is a suitable null vector. It can be easily proved that

$$P^{-1} = \frac{1}{2}\begin{bmatrix} I & J \\ I & -J \end{bmatrix} \quad , \quad \widetilde{P}^{-1} = \frac{1}{2}\begin{bmatrix} I & O & J \\ O & 2 & O \\ I & O & -J \end{bmatrix} .$$

3.1. Let

$$H = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & b & a \\ 0 & 0 & 0 & \cdots & b & a & b \\ 0 & 0 & 0 & \cdots & a & b & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & b & a & \cdots & 0 & 0 & 0 \\ b & a & b & \cdots & 0 & 0 & 0 \\ a & b & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} .$$

If the order of $H$ is $n = 2k$, we can write

$$H = \begin{bmatrix} H_1 & H_2 \\ H_2 & JH_1J \end{bmatrix}$$

(with $H_1$, $H_2$, $J$ of order $k$) and

$$\hat{H} = P^{-1}HP \begin{bmatrix} JH_2+H_1 & O \\ O & -(JH_2-H_1) \end{bmatrix} = \operatorname{diag}\{C_1, -C_2\},$$

where

$$C_1 = \begin{bmatrix} a & b & 0 & \cdots & 0 & 0 \\ b & a & b & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & b & a+b \end{bmatrix}, C_2 = \begin{bmatrix} a & b & 0 & \cdots & 0 & 0 \\ b & a & b & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & b & a-b \end{bmatrix}$$

whose eigenvalues and eigenvectors are known (see 2.2, 2.1). [(1)]

If the order of $H$ is $n = 2k + 1$, we can write

$$H = \begin{bmatrix} O & v & H_2 \\ v^T & a & v^TJ \\ H_2 & Jv & O \end{bmatrix}$$

($H_2, J, O$ are of order $k$, $v$ is a $k$- vector) and

(1) It is well known that if $R = \operatorname{diag}\{S,T\}$ eigenvalues of $R$ are those of $S$ and of $T$. Now, if $\lambda$ and $\mu$ are, in order, eigenvalues of $S$ and $T$, $x_1$ an eigenvector associated to $\lambda$, and $y_1$ an eigenvector associated to $\mu$, then

$$x = \begin{bmatrix} x_1 \\ O \end{bmatrix}, \; y = \begin{bmatrix} O \\ y_1 \end{bmatrix}$$

are eigenvectors of $R$ associated to $\lambda$ and $\mu$, respectively.

$$\hat{H} = \tilde{P}^{-1} H \tilde{P} = \begin{bmatrix} JH_2 & v & O \\ 2v^T & a & O \\ O & O & -JH_2 \end{bmatrix} = \text{diag}\left\{\hat{C}_1, -\hat{C}_2\right\}$$

where $\hat{C}_1$ has order $k+1$, $\hat{C}_2$ has order $k$. Eigenvalues and eigenvectors of matrices

$$\hat{C}_1 = \begin{bmatrix} a & b & 0 & \cdots & 0 & 0 \\ b & a & b & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & 2b & a \end{bmatrix}, \hat{C}_2 = \begin{bmatrix} a & b & 0 & \cdots & 0 & 0 \\ b & a & b & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & b & a \end{bmatrix}$$

are known (see 2.3, 1.4).

3.2. The following matrices are obtained from the matrix $H$ defined in 3.1. They are similar to block-diagonal matrices, whose eigenvalues and eigenvectors are known.

a) Let

$$H_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & b & a+b \\ 0 & 0 & \cdots & b & a & b \\ 0 & 0 & \cdots & a & b & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ b & a & \cdots & 0 & 0 & 0 \\ a+b & b & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

If its order is even, $P^{-1}H_1P = \text{diag}\{T_6, -JT_7J\}$, where $T_6$ and $T_7$ are defined in 2.4 and 2.5 respectively; if its order is odd, $\tilde{P}^{-1}H_1\tilde{P} = \text{diag}\{T_{10}, -T_4\}$, where $T_{10}$ and $T_4$ are defined in 2.8 and 2.2.

b) Let

$$H_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & b & a-b \\ 0 & 0 & \cdots & b & a & b \\ 0 & 0 & \cdots & a & b & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ b & a & \cdots & 0 & 0 & 0 \\ a-b & b & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Then if its order is even, $P^{-1}H_2P = \text{diag}\{T_7, -T_8\}$, while if its order is odd, $\tilde{P}^{-1}H_2\tilde{P} = \text{diag}\{T_{11}, -T_3\}$ where $T_7, T_8, T_{11}, T_3$ have been studied in 2.5, 2.6, 2.9, 2.1 respectively.

c) Let

$$H_3 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 2b & a \\ 0 & 0 & \cdots & b & a & b \\ 0 & 0 & \cdots & a & b & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ b & a & \cdots & 0 & 0 & 0 \\ a & 2b & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

If its order is even, then $P^{-1}H_3P = \text{diag}\{JT_{10}J, -JT_{11}J\}$; if its order is odd, $\tilde{P}^{-1}H_3\tilde{P} = \text{diag}\{T_9, -T_5\}$, where $T_{10}$, $T_{11}$, $T_9$, $T_5$ are defined in 2.8, 2.9, 2.7, 2.3.

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