THEOREMS ON THE FIXED POINT FOR MULTIVALUED MAPPINGS IN TOPOLOGICAL VECTOR SPACES (*)

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SOMMARIO. - *Si dimostra una generalizzazione del teorema di punto fisso di Kakutani in un TV spazio.*

SUMMARY. - *A generalization of the fixed point theorem of Kakutani in TV space is given.*

In the recent time there are many fixed point theorems in not necessarily locally convex topological vector spaces. Such theorems are proved, for example, in [3], [4], [5], [6], [8], [9], [10], [11], [12], [13], [14], [15], [16], [19], [21], [22], [23], [25], [27] and [28].

A bibliography of papers from this field of the fixed point theory can be found in [10].

Since many important topological vector spaces are not locally convex

$$(L^p, 0<p<1, \text{ the Hardy spaces } H^p, 0<p<1, \ (S(X,A,m),d)\ [22])$$

it is of interest to translate the fixed point theory on such spaces. Some useful results in this direction are obtained in [22] with many applications on Hammerstein's equations and in the paper [16] (an application on integral equations). For multivalued mappings some results are obtained in [3], [5], [8], [9], [11], [12], [13], [14], [15].

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An application on sets with convex section (related to minimax problem) is given in [10]. Theorems on almost continuous selection property are proved in [3] and [9].

The aim of this paper is to prove a generalization of a result of Ch.W.Ha, which is in fact a generalization of Kakutani's fixed point theorem.

LEMMA: *Let M be an n-simplex and K be a compact, convex subset of a Hausdorff topological vector space. If q is an upper semicontinuous set valued mapping defined on M such that $q(x)$ is a nonempty closed, convex subset of K, for each $x \in M$ and if $p: K \to M$ is a continuous mapping, then there exists $x_0 \in M$ so that*

$$x_0 \in p(q(x_0)).$$

Remark: If p is a nonlinear mapping then from the convexity of $q(x)$ does not follow the convexity of $p(q(x))$ in general. So this generalization of Kakutani's fixed point theorem can be applied in these cases when the multivalued mapping is not with convex values.

First, we shall give some definitions and notations. In the following text we shall suppose that all topological vector spaces are Hausdorff.

DEFINITION 1: *Let X be a topological vector space, $\mathcal{U}$ the family of neighbourhoods of zero $0 \in X$ and $M \subseteq X$. We say that the set M is of Z type if and only if for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ so that the convex hull $\mathrm{co}(U \cap (M - M))$ is in V.*

An example of such a subset in the space $S(0,1)$ of classes of real measurable finite functions on the interval $[0,1]$ will be given here.

Let us remark that every subset M of a locally convex space X is of Z type since we can suppose that $V = \mathrm{co}\, V$, for every $V \in \mathcal{U}$ and so we can take for a neighbourhood U from the Definition 1 the neighbourhood V.

Now we shall give the definition of a paranormed space which is in general a non locally convex topological vector space. Some fixed point theorems for singlevalued and multivalued mappings in paranormed spaces are proved in [11], [28].

Let E be a linear space over the real or complex number field. The function $\| \ \|^*: E \to [0, \infty)$ will be called a paranorm if:

1. $\| x \|^* = 0 \Leftrightarrow x = 0$.
2. $\| -x \|^* = \| x \|^*$, for every $x \in E$.
3. $\| x + y \|^* \leqslant \| x \|^* + \| y \|^*$, for every $x, y \in E$.
4. If $\| x_n - x_0 \|^* \to 0$ and $\lambda_n \to \lambda_0$ then $\| \lambda_n x_n - \lambda_0 x_0 \| \to 0$.

DEFINITION 2: *The paranormed space* $(E, \| \ \|^*)$ *is a topological vector space in which the topology is defined by the family* $\{V_r\}_{r>0}$ *where:*

$$V_r = \{x \mid x \in E, \|x\|^* < r\}.$$

In [28] Zima has proved a generalization of Schauder's fixed point theorem in paranormed spaces for mapping $f: K \rightarrow K$, where K is a convex, closed and bounded subset of a paranormed space such that:

$\|tx\|^* \leqslant Ct\|x\|^*$, for every $t \in [0,1]$ and every $x \in fK - fK$ and f is completely continuous.

So we shall introduce the following definition.

DEFINITION 3: *Let* $(E, \| \ \|^*)$ *be a paranormed space,* $K \subseteq E$ *and there exists* $C(K) > 0$ *such that for every* $t \in [0,1]$ *and* $x \in K - K$

$$\|tx\|^* \leqslant C(K)\, t\, \|x\|^*.$$

Then we say that the set K *satisfies the Zima condition.*

If $(E, \| \ \|^*)$ is a paranormed space and K is a subset of E which satisfies the Zima condition then for every $r > 0$

$$\mathrm{co}(V_{r/C(K)} \cap (K - K)) \subseteq V_r$$

which can be easily verified. So K is of Z type in the sense of Definition 1.

In [28] an example is given of E and K where $C(K) = 3$. Let $(S(0,1), \| \ \|^*)$ be the paranormed space with the paranorme $\| \ \|^*$ defined by:

$$\|\hat{x}\|^* = \int_0^1 \frac{|x(t)|}{1 + |x(t)|} (dt) \quad \{x(t)\} \in \hat{x} \in S(0,1)$$

and for every $t > 0$ let:

$$K_t = \{\hat{x} \mid \hat{x} \in S(0,1) \text{ and } |x(u)| \leqslant t \text{ for every } u \in [0,1]\}.$$

It is easy to prove that $C(K_t) = 1+2t$, and so the set K_t is for every $t > 0$ of Z type.

The second example is the following. Let E be a vector space over $\mathcal{K}$ (real or complex number field), $\mathbf{R}_\Delta$ be the set of all mappings from Δ into $\mathbf{R}$ with the Tihonov product topology and the operations $+$ and scalar multiplication as usual.

If $f, g \in \mathbf{R}_\Delta$ we say that $f \leqslant g$ if and only if

$$f(t) \leqslant g(t), \text{ for every } t \in \Delta.$$

By $\mathbf{P}_\Delta$ we shall denote the cone of nonnegative elements in $\mathbf{R}_\Delta$. In

[20] Kasahara introduced the notion of a paranormed space (we shall say Φ paranormed space).

DEFINITION 4: *The triplet* $(E, \| \ \|, \Phi)$ *is a* Φ *paranormed space if and only if* $\| \ \| : E \to \mathbf{P}_\Delta, \Phi$ *is a linear, continuous and positive mapping from* $\mathbf{R}_\Delta$ *into* $\mathbf{R}_\Delta$ *such that the following conditions are satisfied:*

1. $\| x \| = 0 \Leftrightarrow x = 0.$
2. $\| tx \| = | t | \, \| x \|$, *for every* $x \in E$ *and every* $t \in \mathcal{K}$.
3. $\| x + y \| \leqslant \Phi(\| x \|) + \Phi(\| y \|)$, *for every* $x, y \in E$.

The topology in $(E, \| \ \|, \Phi)$ is introduced in the following way. Let us denote by $\mathcal{U}$ the family of neighbourhoods of zero in $\mathbf{R}_\Delta$ and for every $U \in \mathcal{U}$ we shall denote the set

$$\{x \mid x \in E, \| x \| \in U\}$$

by V_U.

Then E is a topological vector space in which the family $\{V_U\}_{U \in \mathcal{U}}$ is the fundamental family of neighbourhoods of zero in E. In [20] Kasahara has proved that every Hausdorff topological vector space E is a Φ paranormed space $(E, \| \ \|, \Phi)$ over a topological semifield $\mathbf{R}_\Delta$. In [4] the following definition is given.

DEFINITION 4: *Let* $(E, \| \ \|, \Phi)$ *be a* Φ *paranormed space over a topological semifield* $\mathbf{R}_\Delta$ *and* $K \subseteq E$. *If for every* $n \in N$, *every* $u_i \in K - K$ $(i = 1, 2, \ldots, n)$ *and* $(s_1, s_2, \ldots, s_n) \in [0, 1]^n$ $\sum_{i=1}^{n} s_i = 1$ *the inequality*

$$\| \sum_{i=1}^{n} s_i u_i \| \leqslant \sum_{i=1}^{n} s_i \Phi(\| u_i \|)$$

is satisfied we say that the set K *is of* Φ *type.*

Some fixed point theorems for singlevalued and multivalued mappings which are defined on a subset of Φ type are proved in [3] and [4].

Let us show that every subset $K \subseteq E$ which is of Φ type is also of Z type.

Let $\mathcal{U}' = \{V_U\}_{U \in \mathcal{U}}$ be the family of neighbourhoods of zero in E. If $V \in \mathcal{U}'$ then there exists a subset $\mu = \{t_1, t_2, \ldots, t_n\}$ of Δ and $r > 0$ so that $\| u \| \in U_{\mu, r} \Rightarrow u \in V$ where:

$$U_{\mu, r} = \{x \mid x \in E, \| x \| (t) < r, \text{ for every } t \in \mu\}.$$

Since Φ is linear and continuous mapping there exists $V' \in \mathcal{U}'$ such that $u \in V' \Rightarrow \Phi(\| u \|) \in U_{\mu,r}$.

Then it is easy to see that

$$\mathrm{co}(V' \cap (K - K)) \subseteq V$$

which means that the set K is of Z type.

In the next Theorem we shall denote by $\mathfrak{R}(K)$ the family of all closed and convex subsets of K, and $\mathrm{co}\, M$ is the convex hull of M.

THEOREM 1: *Let X a metrisable topological vector space, Y be a topological vector space, $K = \mathrm{co}\, K, K$ be compact and sequentially compact, $K \subseteq Y, M$ be a compact and convex subset of X, f be an upper semicontinuous mapping from M into $\mathfrak{R}(K)$ and p be a continuous mapping from K into M. If $f(M)$ and $p(\mathrm{co}\, f(M))$ be of Z type then there exists $x_0 \in M$ such that $x_0 \in p(f(x_0))$.*

Proof: Since X is metrisable let $\mathcal{U}_X = \{U_n\}_{n \in N}$ be the fundamental system of neighbourhoods of zero in X and suppose that the family $\{U_n\}_{n \in N}$ is monotone decreasing. For every $n \in N$ let B_n be such finite set that $M \subseteq \bigcup_{b \in B_n} \{x_{n,b} + U_n\}$ and let $\{p_{n,b}\}_{b \in B_n}$ be the partition of the unity subordinated to the covering $\{x_{n,b} + U_n\}_{b \in B_n}$. We suppose that for every $n \in N$ the set U_n is open and balanced. Let, for every $n \in N$ and $b \in B_n, y_{n,b} \in f(x_{n,b})$ and for every $x \in M$:

$$f_n(x) = \sum_{b \in B_n} p_{n,b}(x)\, y_{n,b}\,.$$

If $C_n = \mathrm{co}\{y_{n,b} \mid b \in B_n\}$ then $f_n : M \to C_n$, for every $n \in N$ and the mapping f_n is continuous. Now, let us show that for every $n \in N$ there exists $x_n \in M$ such that $x_n = p \circ f_n(x_n)$. We shall apply Rzepecki's fixed point theorem on the mapping $p \circ f_n : M \to M$ [27]. The mapping $p \circ f_n$ is continuous and $p \circ f_n(M)$ is of Z type. In order to apply Rzepecki's fixed point theorem we must prove that for every $x \in p(\mathrm{co}\, f(M))$ and every $V \in \mathcal{U}_X$ there exists $U \in \mathcal{U}_X$ such that:

$$\mathrm{co}((x + U) \cap p(\mathrm{co}\, f(M))) \subseteq x + V. \tag{1}$$

Let $V \in \mathcal{U}_X$. Since the set $p(\mathrm{co}\, f(M))$ is of Z type there exists $U \in \mathcal{U}_X$ such that

$$\mathrm{co}(U \cap (p(\mathrm{co}\, f(M)) - p(\mathrm{co}\, f(M)))) \subseteq V.$$

Let us show that (1) holds. If $y \in \mathrm{co}((x + U) \cap p(\mathrm{co}\, f(M)))$ it follows that

$$y = \sum_{i=1}^{n} t_i u_i\,, t_i \geqslant 0 (i = 1, 2, \ldots, n),\ \sum_{i=1}^{n} t_i = 1$$

where $u_i \in (x + U) \cap p(\mathrm{co}\, f(M))$. Then $u_i = x + v_i \in p(\mathrm{co}\, f(M))$, where $v_i \in U (i = 1, 2, \ldots, n)$. From this it follows that

$$y = \sum_{i=1}^{n} t_i u_i = \sum_{i=1}^{n} t_i(x + v_i) = x + \sum_{i=1}^{n} t_i v_i$$

and since $v_i = u_i - x \in p(\text{co} f(M)) - p(\text{co} f(M))$ we conclude that $\sum_{i=1}^{n} t_i v_i \in \text{co}(U \cap (p(\text{co} f(M)) - p(\text{co} f(M))))$. So $y \in x + V$.

The mapping $p \circ f_n$ satisfies all the conditions of Rzepecki's fixed point theorem and for every $n \in N$ there exists $x_n \in M$ such that $x_n = p \circ f_n(x_n)$. Let for every $n \in N, u_n = f_n(x_n)$. The set K is compact and so let $u_0 \in K$ be the limit point of the set $\{u_n \mid n \in N\}$. If $x_0 = p(u_0)$ we shall prove that $u_0 \in f(x_0)$. From the relation $p(u_n) = p \circ f_n(x_n) = x_n$, since the mapping p is continuous, it follows that there exists a subsequence $\{x_{n_k}\}_{k \in N}$ of the sequence $\{x_n\}_{n \in N}$ such that $\lim_{k \to \infty} x_{n_k} = x_0$. Here we suppose that $\lim u_n = u_0$.

Let us denote by $\mathcal{U}_Y$ the fundamental system of neighbourhoods of zero in Y. Let V be an arbitrary element from $\mathcal{U}_Y$ and $G_V = f(x_0) + V$. We shall prove that $u_0 \in \bar{G}_V$, for every $V \in \mathcal{U}_Y$.

Let $V \in \mathcal{U}_Y$. Since the set $f(M)$ is of Z type there exists $U \in \mathcal{U}_Y$ such that

$$\text{co}(U \cap (f(M) - f(M))) \subseteq V.$$

The mapping f is upper semicontinuous and so there exists $U_1 \in \mathcal{U}_X$ such that $f(x_0 + U_1) \subseteq G_U$. Let $U_2 \in \mathcal{U}_X$ be such that $U_2 + U_2 \subseteq U_1$ and $k_0 \in N$ such that for every $k \geqslant k_0$ we have:

$$U_{n_k} \subseteq U_2, x_{n_k} \in x_0 + U_2.$$

Then we have the following implication:

$$k \geqslant k_0, b \in B_{n_k}, p_{n_k, b}(x_{n_k}) > 0 \Rightarrow x_{n_k, b} - x_0 \in U_1$$

since $p_{n_k, b}(x_{n_k}) > 0$ implies that $x_{n_k} \in x_{n_k, b} + U_{n_k} \subseteq x_{n_k, b} + U_2$ and so:

$$x_{n_k, b} - x_0 = x_{n_k, b} - x_{n_k} + x_{n_k} - x_0 \in U_2 + U_2 \subseteq U_1.$$

Now, let $k \geqslant k_0$. From $u_n = f_n(x_n)$ for every $n \in N$ we obtain:

$$u_{n_k} = \sum_{b \in B_{n_k}} p_{n_k, b}(x_{n_k}) y_{n_k, b} = \sum_{b \in B_{n_k}, p_{n_k, b}(x_{n_k}) > 0} p_{n_k, b}(x_{n_k}) y_{n_k, b}.$$

Here $y_{n_k, b} \in G_U$ since $x_{n_k', b} \in x_0 + U_1, y_{n_k, b} \in f(x_{n_k, b})$ and $f(x_0 + U_1) \subseteq G_U$. From $y_{n_k, b} \in G_U$ for $p_{n_k, b}(x_{n_k}) > 0$ it can be easily shown that:

$$u_{n_k} \in f(x_0) + \text{co}\,(U \cap (f(M) - f(M))) \subseteq f(x_0) + V$$

for every $k \geqslant k_0$. From this it follows that $u_0 \in f(x_0) + V$ and since V is an arbitrary element from $\mathcal{U}_Y$ we conclude that $u_0 \in f(x_0)$.

Remark: If X and Y are locally convex topological linear space

$W \subset X, G: W \to Y$ is a continuous set-valued function and $f: G(W) \to X$ is a continuous function, under some additional conditions, in [6] Halpern has proved the existence of an element $x \in W$ such that $x \in F(x)$, where $F = f \circ G: W \to X$.

COROLLARY: *Let X and Y be topological vector spaces and X be metrisable, $K = \text{co}\, K, K$ be compact and sequentially compact, $K \subseteq Y, M$ be a compact and convex subset of $X, G: X \to X$ be a linear one to one mapping such that G and G^{-1} be continuous, f be an upper semicontinuous mapping from M into $\mathfrak{R}(K), p: K \to M$ be a continuous mapping such that $p(K) \subseteq G(M)$ and the sets $f(M)$ and $p(\text{co}\, f(M))$ be of Z type. Then there exists $x_0 \in M$ such that*

$$G(x_0) \in p(f(x_0)).$$

Proof: We shall prove that there exists $x_0 \in M$ such that

$$x_0 \in G^{-1} p(f(x_0)).$$

Since the mapping $G^{-1} p$ is continuos and $p(K) \subseteq G(M)$ it remains to show that the set $G^{-1} p(\text{co}\, f(M))$ satisfies the following condition:

If $\mathcal{U}_X$ is the fundamental system of neighbourhoods of zero in X then for every $V \in \mathcal{U}_X$ there exists $U \in \mathcal{U}_X$ such that $\text{co}(U \cap (G^{-1} p(\text{co}\, f(M)) - G^{-1} p(\text{co}\, f(M)))) \subseteq V$.

Let $V \in \mathcal{U}_X$. Since the mapping G^{-1} is continuous there exists $V' \in \mathcal{U}_X$ so that $G^{-1} V' \subseteq V$. The set $p(\text{co}\, f(M))$ is of Z type and so there exists $U' \in \mathcal{U}_X$ so that

$$\text{co}(U' \cap (p(\text{co}\, f(M)) - p(\text{co}\, f(M)))) \subseteq V'.$$

The mapping G^{-1} is linear and so it follows that

$$G^{-1}(\text{co}(U' \cap (p(\text{co}\, f(M)) - p(\text{co}\, f(M))))) =$$
$$= \text{co}(G^{-1}(U' \cap (p(\text{co}\, f(M)) - p(\text{co}\, f(M))))).$$

If $U \in \mathcal{U}_X$ is such that $G(U) \subseteq U'$ we have that

$$\text{co}(U \cap (G^{-1} p(\text{co}\, f(M)) - G^{-1} p(\text{co}\, f(M)))) \subseteq V$$

and so all the conditions of Theorem 1 are satisfied which implies that there exists $x_0 \in M$ such that $x_0 \in G^{-1} p(f(x_0))$ and $G(x_0) \in p(f(x_0))$.

Now, we shall prove a Proposition about the almost continuous selection property. This property is related with fixed point theory [1].

DEFINITION 5: *Let X and Y be topological vector spaces, $M \subseteq X, K \subseteq Y, f: M \to 2^K$ and $\mathcal{U}$ be the fundamental system of neighbourhoods of zero in Y. If for every $V \in \mathcal{U}$ there exists a continuous mapping $g_V: M \to K$ such that $g_V(x) \in f(x) + V$, for every $x \in M$ we say that the mapping f has the almost continuous selection property.*

Some interesting theorems about the almost continuous selection property are proved by Michael in [10]. In the following definition we shall denote by $\mathcal{U}_X$ the fundamental system of neighbourhoods of zero in topological vector space X and by $\mathcal{U}_Y$ the fundamental system of neighbourhoods of zero in topological vector space Y. If $V \in \mathcal{U}_Y$ and $A \subseteq Y$ we shall write

$$V[A] = \{y \mid y \in Y, \text{ there exists } z \in A \text{ such that } z - y \in V\}.$$

In [3] the following Definition is given, where X and Y are topological vector spaces.

DEFINITION 6: *A multivalued mapping $f: M \to 2^K (M \subseteq X, K \subseteq Y)$ is u-continuous iff for every $V \in \mathcal{U}_Y$ there exists $W \in \mathcal{U}_X$ such that $x_1 - x_2 \in W (x_1, x_2 \in M)$ implies $f(x_1) \subseteq V[f(x_2)]$ and $f(x_2) \subseteq V[f(x_1)]$.*

The following Proposition is a generalization of Theorem 1 from [2].

PROPOSITION: *Let X and Y be topological vector spaces, M be a paracompact convex subset of X, K be a convex and compact subset of $Y, f: M \to \mathcal{R}(K)$ be an u-continuous mapping, $p: K \to M$ be a continuous mapping such that $f(M)$ is of Z type. Then the mapping $p \circ f$ has the almost continuous selection property.*

Proof: Let V be an arbitrary element from $\mathcal{U}_X$. Since K is compact and p is continuous there exists $V' \in \mathcal{U}_Y$ such that

$$p((y + V') \cap K) \subseteq p(y) + V, \text{ for every } y \in K.$$

Further, the set $f(M)$ is of Z type and so there exists $U \in \mathcal{U}_Y$ such that

$$\text{co}(U \cap (f(M) - f(M))) \subseteq V'.$$

Since the mapping f is u-continuous there exists $W(U) \in \mathcal{U}_X$ such that the following implication holds:

For every $x_1, x_2 \in M, x_1 - x_2 \in W, y_1 \in f(x_1) \Rightarrow$ there exists $y_2 \in f(x_2)$ such that $y_1 - y_2 \in U$.

The set M is paracompact and so there exists a locally finite partition Q of the unity subordinated to the open cover $\{x + W\}_{x \in M}$ (we shall suppose that every element from $\mathcal{U}_X$ is open and simmetric). From this we conclude that there exists a mapping $h: Q \to M$ such that $q(x) = 0$, for every $x \in M \setminus \{h(q) + W\}$.

Now, we shall define, similarly as in [2], $g_V: M \to K$ in the following way:

$$g_V(x) = \sum_{q \in Q} q(x)\, z(h(q)), \text{ for all } x \in M$$

where z is a chois function for the family $\{f(x)\}_{x \in M}$. If

$$\mathfrak{M}(x) = \{q \mid q \in Q, q(x) \neq 0\}, \; x \in M$$

then $\mathfrak{M}(x)$ is a finite subset of Q and from $q \in \mathfrak{M}(x)$ it follows that $x \in h(q) + W$ and so $h(q) - x \in W$, since W is symmetric.

The mapping f is u-continuous and since $z(h(q)) \in f(h(q))$ we conclude that there exists $v_q(x) \in f(x)$ so that $z(h(q)) - v_q(x) \in U$.

Further let for every $x \in M$ and $q \in Q$

$$u_q(x) = \begin{cases} v_q(x) & q \in M(x) \\ z(x) & q \in Q \setminus M(x) \end{cases}$$

and $s(x) = \sum_{q \in Q} q(x)\, u_q(x)$. It is obvious that $s(x) \in f(x)$.

From the relation $\mathrm{co}(U \cap (f(M) - f(M))) \subseteq V'$ it follows that

$$g_V(x) - s(x) = \sum_{q \in M(x)} q(x)\,[z(h(q)) - v_q(x))] \in$$

$$\in \mathrm{co}(U \cap (f(M) - f(M))) \subseteq V'$$

and so for every $x \in M$ we have that $g_V(x) \in s(x) + V'$. From this we obtain that

$$p(g_V(x)) \in p((s(x) + V') \cap K), \text{ for every } x \in M.$$

Since

$$p((y + V') \cap M) \subseteq p(y) + V, \text{ for every } y \in M$$

it follows that

$$p((s(x) + V') \cap K) \subseteq p(s(x)) + V, \text{ for every } x \in M.$$

So the mapping $p \circ f$ has the almost continuous selection property since $p \circ g_V$ is a continuous mapping from M into M.

Similarly as in [1] we have the following Lemma.

LEMMA: *Let X be a topological vector space and C be a nonempty compact and convex subset of X. Let $f : C \to 2^C$ be a closed multivalued mapping which has the almost continuous selection property and $\mathcal{U}$ be the fundamental system of neighbourhoods of zero in X. If for every $V \in \mathcal{U}$, $f_V : C \to C$ is such that $f_V(x) \in f(x) + V$, for every $x \in C$ and $f_V(C)$ is of Z type then there exists a fixed point for the mapping f.*

Proof: From Rzepecki's fixed point theorem it follows that there exists, for every $V \in \mathcal{U}$, $x_V \in C$ so that

$$x_V = f_V(x_V)$$

which implies that $x_V \in f(x_V) + V$. Since C is compact and f is closed as in [1] it follows the Lemma.

Now, it is easy to prove the following fixed point theorem.

THEOREM 2: *Let X and Y be topological vector spaces, M be a nonempty compact and convex subset of X, K be a convex and compact subset of Y, $f: M \to \mathfrak{R}(K)$ be a closed u-continuous mapping, $p: K \to M$ be a continuous mapping such that $f(M)$ is of Z type and $p(\text{co} f(M))$ is of Z type. Then there exists $x_0 \in M$ such that $x_0 \in p(f(x_0))$.*

Proof: From the Proposition it follows that the mapping $p \circ f$ has the almost continuous selection property and that there exists for every neighbourhood V in X, $(p \circ f)_V = p(g_V)$ so that:

$$(p \circ f)_V(x) \in (p \circ f)(x) + V, \text{ for every } x \in M$$

and that (see the proof of the Proposition) $(pf)_V(M) \subseteq p(\text{co}(M))$. Since $p(\text{co} f(M))$ is of Z type it follows that $(pf)_V$ is of Z type for every V and from the Lemma it follows that there exists x_0 so that $x_0 \in p(f(x_0))$.

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