ON DISTAL SPACES (*)

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SOMMARIO. - *Data una struttura distal* **D** *su un insieme S, si ottiene in modo naturale una topologia* $\mathcal{T}_{\mathbf{D}}$ *su S. Si esaminano poi le relazioni fra strutture distal, topologie, prossimità, uniformità e metriche.*

SUMMARY. - *Any distal structure* **D** *on a set S induces in S a topology* $\mathcal{T}_{\mathbf{D}}$ *in a natural way. We examinate the relations between distal structures, topologies, proximities, uniformities and metrics.*

1. The topology of a distal space.

DEFINITION 1. (1) - *A non void collection* **D** *of families of subsets*

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(1) See [2]. Conditions (D_1), (D_2), (D_3) hold iff (see [5], Theorem 3):

(1) $\{A_i\}$ $(i \in I) \in \mathbf{D} \Rightarrow A_i \cap A_j = \emptyset$ whenever $i \neq j$;

(2) $X \subseteq S \Rightarrow \{X\} \in \mathbf{D}$;

(3) if $\mathcal{A} \in \mathbf{D}$ and $\mathcal{B} \in \mathbf{D}$, then $\mathcal{A} \wedge \mathcal{B} \in \mathbf{D}$;

(4) if $\mathcal{A} = \{A_i\}$ $(i \in I) \in \mathbf{D}$ and $\gamma = \{I_\lambda\}$ $(\lambda \in \Lambda)$ is any partition of I, then $\mathcal{A}_{/\gamma} \in \mathbf{D}$ (where $\mathcal{A}_{/\gamma} = \{A_\lambda\}$ $(\lambda \in \Lambda)$ and $A_\lambda = \bigcup_{i \in I_\lambda} A_i$);

(5) if $\mathcal{A} = \{A_i\}$ $(i \in I)$ is a family of subsets of S, and if X and Y are subsets of S such that $\{X - Y, Y - X\} \in \mathbf{D}$, $\mathcal{A}_{/X} \in \mathbf{D}$ and $\mathcal{A}_{/Y} \in \mathbf{D}$ (where $\mathcal{A}_{/Z} = \{A_i \cap Z\}$ $(i \in I)$ for any $Z \subseteq S$), then $\mathcal{A}_{/X \cup Y} \in \mathbf{D}$;

(6) if $\mathcal{A} = \{A_i\}(i \in I) \in \mathbf{D}$, then there is a family $\mathcal{B} = \{B_i\}(i \in I) \in \mathbf{D}$ such that $A_i \subseteq B_i$ for each $i \in I$ and moreover $\{\bigcup_{i \in I} A_i, S - \bigcup_{i \in I} B_i\} \in \mathbf{D}$.

of a non empty set S is a distal structure on S if:

(D_1) $\{A, B\} \in \mathbf{D} \Rightarrow A \cap B = \emptyset$;

(D_2) *any distal combination of a finite number of elements of* **D** *belongs to* **D**;

(D_3) *any element of* **D** *has a distal neighbourhood (written distal nbd) in* **D**.

DEFINITION 2. - *Let* $(S, \mathbf{D})$ *be a distal space and* $x \in S$. *A subset* U *of* S *is a* **D**-*neighbourhood (written* **D**-*nbd) of* x *iff* $\{\{x\}, S - U\} \in \mathbf{D}$.

PROPOSITION 1. - *Let* $(S, \mathbf{D})$ *be a distal space. There exists a topology* $\mathcal{T}_{\mathbf{D}}$ *on* S, *such that, for any* $x \in S$, *the collection* $\mathcal{U}_{\mathbf{D}}(x)$ *of all* **D**-*nbds of* x *is the neighbourhood filter of* x.

Proof. Clearly $\mathcal{U}_{\mathbf{D}}(x)$ is a filter and $\overset{\frown}{x} \leqslant \mathcal{U}_{\mathbf{D}}(x)$. If $U \in \mathcal{U}_{\mathbf{D}}(x)$ and $\{V, W\}$ is a distal nbd of $\{\{x\}, S - U\}$ in **D**, then $\{\{y\}, S - U\} \in \mathbf{D}$ for any $y \in V$.

DEFINITION 3. - *The topology* $\mathcal{T}_{\mathbf{D}}$ *from Proposition 1 will be called the distal topology of* $(S, \mathbf{D})$ *or* **D**-*topology.*

REMARK 1. - Let $(S, \mathbf{D})$ be a distal space and $X \subseteq S$. Then:
the set $\mathring{X} = \{x \in S \,/\, \{\{x\}, S - X\} \in \mathbf{D}\}$ is the interior of X;
the set $\bar{X} = \{x \in S \,/\, \{\{x\}, X\} \notin \mathbf{D}\}$ is the closure of X.

PROPOSITION 2. - *Let* $(S, \mathbf{D})$ *be a distal space and*

$$\mathcal{A} = \{A_i\}\ (i \in I) \in \mathbf{D}.$$

If $\mathcal{B} = \{B_i\}\ (i \in I)$ *is a distal nbd of* $\mathcal{A}$ *in* **D**, *then:*

(1) *Each* B_i *is a neighbourhood of* A_i;

(2) $\overline{\bigcup_{j \in J} A_j} \subseteq \bigcup_{j \in J} B_j$ *for any* $J \subseteq I$.

Proof. Ad (1). Let $j \in I$ and $x \in A_j$. By (D_2) we obtain $\{\{x\}, S - \bigcup_{i \in I} B_i\} \in \mathbf{D}$ from $\mathcal{A} \vee \{S - \bigcup_{i \in I} B_i\} \in \mathbf{D}$, and $\{\{x\}, \bigcup_{i \neq j} B_i\} \in \mathbf{D}$ from $\mathcal{B} \in \mathbf{D}$.

Ad (2). Given $J \subseteq I$, we have $\overline{(\bigcup_{j \in J} A_j)} \cap (S - \bigcup_{i \in I} B_i) = \emptyset$ since $\{\bigcup_{j \in J} A_j, S - \bigcup_{i \in I} B_i\} \in \mathbf{D}$. If $J = I$, the proof is complete. If $J \subset I$, also $\{\bigcup_{j \in J} A_j, \bigcup_{i \in I - J} B_i\} \in \mathbf{D}$, and so $\overline{(\bigcup_{j \in J} A_j)} \cap (\bigcup_{i \in I - J} B_i) = \emptyset$.

COROLLARY. - *Let* $(S, \mathbf{D})$ *be a distal space. If* $\{A_i\}$ $(i \in I) \in \mathbf{D}$, *also* $\{\overline{A_i}\}$ $(i \in I) \in \mathbf{D}$.

DEFINITION 4. - *Let* $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *be distal structures on a set* S. *We say that* $\mathbf{D}_1$ *is finer than* $\mathbf{D}_2$ *or that* $\mathbf{D}_2$ *is coarser than* $\mathbf{D}_1$, *and we write* $\mathbf{D}_1 \leq \mathbf{D}_2$, *iff* $\mathbf{D}_1 \supseteq \mathbf{D}_2$.

DEFINITION 5. (See [1], [2], [5]) - *Let* $(S, \mathbf{D})$ *and* $(S', \mathbf{D}')$ *be distal spaces. A function* $f: S \to S'$ *is a distal mapping iff* $\{A'_i\}$ $(i \in I) \in \mathbf{D}'$ *implies* $\{f^{-1}(A'_i)\}(i \in I) \in \mathbf{D}$. $(S, \mathbf{D})$ *and* $(S', \mathbf{D}')$ *are distally isomorphic if there is a bijection* $f: S \to S'$, *such that both* f *and* f^{-1} *are distal mappings.*

REMARK 2. - Any distal mapping from $(S, \mathbf{D})$ to $(S', \mathbf{D}')$ is a continuous map from $(S, \mathcal{T}_{\mathbf{D}})$ to $(S', \mathcal{T}_{\mathbf{D}'})$. If $(S, \mathbf{D})$ and $(S', \mathbf{D}')$ are distally isomorphic, then $(S, \mathcal{T}_{\mathbf{D}})$ and $(S', \mathcal{T}_{\mathbf{D}'})$ are homeomorphic.

2. Separation properties of a distal space.

PROPOSITION 3. - *Any distal space* $(S, \mathbf{D})$ *is* T_{3a}.

Proof. Given a closed subset C of $(S, \mathcal{T}_{\mathbf{D}})$ and a point $x \in S - C$, consider the set J_2 of all rational numbers r of form $p/2^n$, $0 \leq p/2^n \leq 1$.

With a process which is similar to the one of Urysohn's Theorem we construct a family $\{A_r\}(r \in J_2)$ of subsets of S, such that:

(1) $\overline{\{x\}} \subseteq A_r \subseteq S - C$ for each $r \in J_2$, and A_r is open if $r \neq 0$;

(2) $\{\overline{A_r}, S - A_s\} \in \mathbf{D}$ whenever $r < s$.

Then we define a continuous map $f: (S, \mathcal{T}_{\mathbf{D}}) \to [0,1]$ putting:

$$f(y) = \begin{cases} 1 & \text{if } y \in C; \\ \inf \Lambda, \text{ where } \Lambda = \{r \in J_2 / y \in A_r\}, & \text{if } \ y \in S - C. \end{cases}$$

PROPOSITION 4. - *Let* $(S, \mathbf{D})$ *be a distal space. If* $(S, \mathcal{T}_{\mathbf{D}})$ *is* T_0, then:

(D_4) $x \neq y \Rightarrow \{\{x\}, \{y\}\} \in \mathbf{D}$.

Conversely, if $\mathbf{D}$ *satisfies* (D_4), *then* $(S, \mathcal{T}_{\mathbf{D}})$ *is Hausdorff.*

3. Distal structures on a topological space.

LEMMA 1. - *Given a topological space $(S, \mathfrak{T})$, let $\mathbf{D}(\mathfrak{T})$ be the collection of all families $\{X_i\}(i \in I)$ of subsets of S such that:*

(i) the set $I' = \{i \in I \,/\, X_i \neq \emptyset\}$ is finite;

(ii) if $\overline{\overline{I'}} \geq 2$, then for each $j \in I'$ there is a continuous map $f_j : (S, \mathfrak{T}) \to [0,1]$ such that $f_j(X_j) = \{1\}$ and $f_j(\bigcup_{i \neq j} X_i) = \{0\}$.

Then $\mathbf{D}(\mathfrak{T})$ is a distal structure on the set S, and the $\mathbf{D}(\mathfrak{T})$-topology is coarser than $\mathfrak{T}$.

Proof.

a) $\mathbf{D}(\mathfrak{T})$ satisfies the conditions from footnote [(1)].

(1) and (2) follow immediately from the definition of $\mathbf{D}(\mathfrak{T})$.

(3). Let $\{A_i\}(i \in I) \in \mathbf{D}$, $\{B_j\}(j \in J) \in \mathbf{D}$, $I' = \{i \in I \,/\, A_i \neq \emptyset\}$, $J' = \{j \in J \,/\, B_j \neq \emptyset\}$, $H = \{(i,j) \in I \times J / A_i \cap B_j \neq \emptyset\}$. If $\overline{\overline{H}} \geq 2$, for any $(i_0, j_0) \in H$ we find two continuous maps f_0 and g_0 from $(S, \mathfrak{T})$ to $[0,1]$, such that $f_0(A_{i_0}) = \{1\}$, $f_0(\bigcup_{i \neq i_0} A_i) = \{0\}$, $g_0(B_{j_0}) = \{1\}$, $g_0(\bigcup_{j \neq j_0} B_j) = \{0\}$. We obtain a continuous map $h_0 : (S, \mathfrak{T}) \to [0,1]$ such that $h_0(A_{i_0} \cap B_{j_0}) = \{1\}$ and $h_0(\bigcup_{(i,j) \neq (i_0, j_0)} (A_i \cap B_j)) = \{0\}$ in the following way. If $I' = \{i_0\}$, we take $h_0 = g_0$; if $J' = \{j_0\}$, we take $h_0 = f_0$; if both $\overline{\overline{I'}} \geq 2$ and $\overline{\overline{J'}} \geq 2$, we put $h_0(x) = f_0(x) \cdot g_0(x)$ for each $x \in S$.

(4). Given a family $\mathfrak{A} = \{A_i\}(i \in I)$ of subsets of S, take $X \subseteq S$ and $Y \subseteq S$, such that $X - Y \neq \emptyset$, $Y - X \neq \emptyset$, $\{X - Y, Y - X\} \in \mathbf{D}(\mathfrak{T})$, $\mathfrak{A}_{/X} \in \mathbf{D}(\mathfrak{T})$, $\mathfrak{A}_{/Y} \in \mathbf{D}(\mathfrak{T})$. Put $I_1 = \{i \in I / A_i \cap X \neq \emptyset\}$, $I_2 = \{i \in I / A_i \cap Y \neq \emptyset\}$, and assume $\overline{\overline{I_1 \cup I_2}} \geq 2$. Let φ and ψ be continuous maps from $(S, \mathfrak{T})$ to $[0,1]$ such that $\varphi(X - Y) = \{1\}$, $\varphi(Y - X) = \{0\}$, $\psi(X - Y) = \{0\}$, $\psi(Y - X) = \{1\}$. Then, if $j \in I_1$ and $\overline{\overline{I_1}} \geq 2$, take a continuous map $f_j : (S, \mathfrak{T}) \to [0,1]$ such that $f_j(A_j \cap X) = \{1\}$ and $f_j(\bigcup_{i \neq j} (A_i \cap X)) = \{0\}$; if $j \in I_2$ and $\overline{\overline{I_2}} \geq 2$, take a continuous map $g_j : (S, \mathfrak{T}) \to [0,1]$ such that $g_j(A_j \cap Y) = \{1\}$ and $g_j(\bigcup_{i \neq j} (A_i \cap Y)) = \{0\}$. for each $j \in I_1 \cup I_2$ we define a continuous map $h_j : (S, \mathfrak{T}) \to [0,1]$ in the following way. If $\overline{\overline{I_1}} \geq 2$, $\overline{\overline{I_2}} \geq 2$ and $j \in I_1 \cap I_2$, we put

$$h_j(x) = \inf\{1, [(f_j(x) + \psi(x)) \cdot (g_j(x) + \varphi(x)) - \varphi(x) \cdot \psi(x)]\}$$

for each $x \in S$. We take $h_j = \varphi$ if $I_1 = \{j\}$ and $j \notin I_2$; $h_j = \psi$ if $I_2 = \{j\}$ and $j \notin I_1$; $h_j = \sup(\varphi, g_j)$ if $I_1 = \{j\}$ and $j \in I_2$; $h_j = \sup(\psi, f_j)$ if

$I_2=\{j\}$ and $j \in I_1$; $h_j=\inf(\varphi, f_j)$ if $j \in I_1-I_2$ and $\overline{\overline{I}}_1\geq 2$; $h_j=\inf(\psi, g_j)$ if $j \in I_2-I_1$ and $\overline{\overline{I}}_2 \geq 2$.

(5). Given $\{A_i\}(i \in I) \in \mathbf{D}(\mathcal{T})$ and a partition $\{I_\lambda\}(\lambda \in \Lambda)$ of I, put $\Lambda' = \{\lambda \in \Lambda / A_\lambda \neq \emptyset\}$ where $A_\lambda = \bigcup_{i \in I_\lambda} A_i$, and assume $\overline{\overline{\Lambda}}' \geq 2$. Then, given $\lambda \in \Lambda'$, consider $I'_\lambda = \{i \in I_\lambda / A_i \neq \emptyset\} = \{i_1, i_2, \dots, i_n\}$. For each $i_r \in I'_\lambda$ take a continuous map $f_r : (S, \mathcal{T}) \to [0,1]$ such that $f_r(A_{i_r}) = \{1\}$ and $f_r(\bigcup_{i \in I-\{i_r\}} A_i) = \{0\}$. The map $f_\lambda : (S, \mathcal{T}) \to [0,1]$, given by $f(x) = \inf\{1, \sum_{i=1}^{n} f_r(x)\}$ for any $x \in S$, is continuous.

(6). Let $\mathcal{A} = \{A_i\}(i \in I) \in \mathbf{D}(\mathcal{T})$ and $I' = \{i \in I / A_i \neq \emptyset\}$. To construct $\mathcal{B} = \{B_i\}(i \in I)$, put $B_i = \emptyset$ for $i \in I - I'$. If $I' = \{i_0\}$, put $B_{i_0} = S$. Now assume $\overline{\overline{I}}' \geq 2$. For each $j \in I'$, take a continuous map $f_j : (S, \mathcal{T}) \to [0,1]$ such that $f_j(\bigcup_{i \neq j} A_i) = \{0\}$, and put

$$B_j = f_j^{-1}(]2/3, 1]) \cap (\bigcap_{i \in I'-\{j\}} f_i^{-1}([0, 1/3[)).$$

Clearly $A_j \subseteq B_j$. $\mathcal{B} \in \mathbf{D}(\mathcal{T})$ since, for each $j \in I'$, the map $g_j : (S, \mathcal{T}) \to [0,1]$, given by $g_j(x) = \sup\{0, \{\inf\{1, 3f_j(x) - 1\}\}$ for any $x \in S$, is continuous. Then for each $j \in I'$ define a map $h_j : S \to [0,1]$, putting $h_j(x) = \sup\{0, 3f_j(x) - 2\}$ for any $x \in S$. $\{\bigcup_{i \in I} A_i, S - \bigcup_{i \in I} B_i\} \in \mathbf{D}(\mathcal{T})$, because the maps h and k from $(S, \mathcal{T})$ to $[0,1]$, given by $h(x) = \inf\{1, \sum_{j \in I'} h_j(x)\}$ and $k(x) = 1 - h(x)$ for any $x \in S$, are continuous.

b) Let $\mathcal{T}'$ be the $\mathbf{D}(\mathcal{T})$-topology in S. $\mathcal{T}'$ is coarser than $\mathcal{T}$, since any closed subset of $(S, \mathcal{T}')$ is closed in $(S, \mathcal{T})$.

DEFINITION 6. - *Let $(S, \mathcal{T})$ be a topological space. A distal structure $\mathbf{D}$ on S is compatible with $\mathcal{T}$ if the $\mathbf{D}$-topology is $\mathcal{T}$.*

THEOREM 1. - *Let $(S, \mathcal{T})$ be a topological space. The distal structure $\mathbf{D}(\mathcal{T})$ from Lemma 1 is compatible with $\mathcal{T}$ iff $(S, \mathcal{T})$ is T_{3a}.*

Proof. The condition is necessary by Proposition 3. Conversely, if $(S, \mathcal{T})$ is T_{3a} and $x \in S$, any open neighbourhood of x in $(S, \mathcal{T})$ is a $\mathbf{D}$-nbd of x.

REMARK 3. - Generally, if $\mathcal{T}$ and $\mathcal{T}'$ are topologies on a set S and $\mathcal{T} \leq \mathcal{T}'$, then $\mathbf{D}(\mathcal{T}) \leq \mathbf{D}(\mathcal{T}')$.

REMARK 4. - Let $(S, \mathcal{T})$ be a topological space. If

$$\{X_i\}(i \in I) \in \mathbf{D}(\mathcal{T}),$$

then $\overline{X_{i_1}} \cap \overline{X_{i_2}} = \emptyset$ whenever $i_1 \neq i_2$.

THEOREM 2. - *Let $(S, \mathcal{T})$ be a topological space and* **D** *the collection of all families $\{X_i\}(i \in I)$ of subsets of S such that:*

(i) the set $I' = \{i \in I / X_i \neq \emptyset\}$ is finite;

(ii') $\overline{X_{i_1}} \cap \overline{X_{i_2}} = \emptyset$ whenever $i_1 \neq i_2$.

Then, if $(S, \mathcal{T})$ is T_3 and T_4, we have $\mathbf{D}(\mathcal{T}) = \mathbf{D}$ and $\mathcal{T}_{\mathbf{D}} = \mathcal{T}$. Conversely, if $\mathbf{D} = \mathbf{D}(\mathcal{T})$, the space $(S, \mathcal{T})$ is T_4.

Proof. Clearly $(S, \mathcal{T})$ is T_4 if $\mathbf{D} = \mathbf{D}(\mathcal{T})$.

Now assume that $(S, \mathcal{T})$ is T_3 and T_4. If $\{X_i\} \in \mathbf{D}$, $\overline{\overline{I'}} \geq 2$ and $j \in I'$, by T_4 there is a continuous map $f_j : (S, \mathcal{T}) \to [0,1]$ such that $f_j(\overline{X_j}) = \{1\}$ and $f(\overline{\bigcup_{i \neq j} X_i}) = \{0\}$; hence $\{X_i\}(i \in I) \in \mathbf{D}(\mathcal{T})$. So $\mathbf{D} = \mathbf{D}(\mathcal{T})$, because $\mathbf{D}(\mathcal{T}) \subseteq \mathbf{D}$ by Remark 4. Then $\mathcal{T}_{\mathbf{D}} = \mathcal{T}$, since $(S, \mathcal{T})$ is T_{3a}.

LEMMA 2. - *Let $(S, \mathbf{D})$ be a χ_0-quasi compact distal space* (2). *Then, if $\{A_i\}(i \in I) \in \mathbf{D}$, the set $I' = \{i \in I / A_i \neq \emptyset\}$ is finite.*

Proof. Assume $\{A_i\}(i \in I) \in \mathbf{D}$ and $\overline{\overline{I'}} \geq \chi_0$. For each $i \in I'$ fix a point $x_i \in A_i$. The set $X = \{x_i\}(i \in I)$ has a cluster point x in S. If $\{B_i\}(i \in I)$ is a distal nbd of $\{A_i\}(i \in I)$ in **D**, we have $x \in \bigcup_{i \in I'} B_i$, because $x \in \overline{\bigcup_{i \in I'} A_i}$; so $x \in B_j$ for exactly one index $j \in I'$. But x is also a cluster point of the set $X - \{x_j\}$; so $x \in \overline{\bigcup_{i \neq j} A_i}$, and hence $x \in \bigcup_{i \neq j} B_j$. Since this is impossible, I' must be finite.

THEOREM 3. - *Let $(S, \mathcal{T})$ be a quasi compact T_3 topological space and* **D** *the collection of all families $\{X_i\}(i \in I)$ of subsets of S such that:*

(i) the set $I' = \{i \in I / X_i \neq \emptyset\}$ is finite;

(ii') $\overline{X_{i_1}} \cap \overline{X_{i_2}} = \emptyset$ whenever $i_1 \neq i_2$.

Then **D** *is the only distal structure on S compatible with $\mathcal{T}$.*

Proof. **D** is a distal structure on S and $\mathcal{T}_{\mathbf{D}} = \mathcal{T}$ by Theorem 2. Now let $\mathbf{D}'$ be a distal structure on S compatible with $\mathcal{T}$. $\mathbf{D}' \subseteq \mathbf{D}$ by Lemma 2 and the Corollary of Proposition 2. Then we easily obtain $\{X, Y\} \in \mathbf{D}'$ whenever X and Y are disjoint closed subsets of $(S, \mathcal{T})$; hence $\mathbf{D} \subseteq \mathbf{D}'$ by (D_2).

(2) χ stands for the hebrew letter «aleph».

COROLLARY 1. - *Let $(S,\mathcal{T})$ and $(S,\mathcal{T}')$ be topological spaces. If S' is quasi compact and T_3, then any continuous map from S to S' is a distal mapping from $(S,\mathbf{D}(\mathcal{T}))$ to $(S',\mathbf{D}')$, where $\mathbf{D}'$ is the distal structure of S'.*

COROLLARY 2. - *Let $(S,\mathcal{T})$ be a T_{3a} topological space. Then $\mathbf{D}(\mathcal{T})$ is the coarsest distal structure on S compatible with $\mathcal{T}$ and such that any continuous map from S to $[0,1]$ is a distal mapping.*

4. Distal structures, proximities, uniformities and metrics.

If $(S,\mathcal{U})$ is a uniform space, the collection $\mathbf{D}(\mathcal{U})$ of all uniformly discrete families is a distal structure on S (see [1]). Moreover:

PROPOSITION 5. - *The $\mathbf{D}(\mathcal{U})$-topology is the uniform topology of $(S,\mathcal{U})$.*

REMARK 5. - Let $(S,\mathbf{D})$ be a distal space and $\mathcal{F}=\{X_i\}(i\in I)\in\mathbf{D}$. The coverings $\mathcal{R}(\mathcal{F})=\{\mathfrak{X}_j\}(j\in I)$, where $\mathfrak{X}_j=X_j\cup(S-\bigcup_{i\in I}X_i)$, form a subbasis of a Tukey-uniformity on S (see [2]). The sets $U(\mathcal{F})=$ $=S\times S-(\bigcup_{i\neq j}(X_i\times X_j))$ form a subbasis for a Weil-uniformity on S (see [5]). These uniformities give the same uniform structure on S, that we denote by $\mathcal{U}(\mathbf{D})$. Since the collection of all uniformly discrete families of $(S,\mathcal{U}(\mathbf{D}))$ is $\mathbf{D}$ (see [2], [5]), the topology of $(S,\mathcal{U}(\mathbf{D}))$ is the $\mathbf{D}$-topology.

Let $(S,\mathcal{T})$ be a T_{3a} topological space. The sets

$$W_{f,r}=\{(x,y)\in S\times S\,/\,|f(x)-f(y)|<r\}$$

(where $r>0$ and $f:S\to[0,1]$ is a continuous map) form a subbasis of a Weil-uniformity on S, which is compatible with $\mathcal{T}$ (see [3], 33.6). Let $\mathcal{U}(\mathcal{T})$ denote such a uniformity. We have:

PROPOSITION 6. - *The uniform structure $\mathcal{U}(\mathcal{T})$ is finer than $\mathcal{U}(\mathbf{D}(\mathcal{T}))$; the distal structure $\mathbf{D}(\mathcal{T})$ is coarser than $\mathbf{D}(\mathcal{U}(\mathcal{T}))$.*

Proof. Let $\mathcal{A}=\{A_i\}(i\in I)\in\mathbf{D}(\mathcal{T})$ and $J=\{i\in I/A_i\neq\varnothing\}$. *If* $\overline{\overline{J}}\geq 2$, for each $j\in J$ take a continuous map $f_j:S\to[0,1]$ such that $f_j(A_j)=\{1\}$ and $f_j(\bigcup_{i\neq j}A_i)=\{0\}$.

a) $\mathcal{U}(\mathcal{T})\leq\mathcal{U}(\mathbf{D}(\mathcal{T}))$. If $\overline{\overline{J}}<2$, $U(\mathcal{A})=S\times S$. If $\overline{\overline{J}}\geq 2$, we have $W_{f_j,1}\cap(\bigcup_{i\neq j}(A_i\times A_i))=\varnothing$ for any $j\in J$; hence $\bigcap_{j\in J}W_{f_j,1}\subseteq U(\mathcal{A})$.

b) $\mathbf{D}(\mathcal{U}(\mathcal{T})) \leq \mathbf{D}(\mathcal{T})$. Clearly $\mathcal{A} \in \mathbf{D}(\mathcal{U}(\mathcal{T}))$ if $\bar{\bar{J}} < 2$. If $\bar{\bar{J}} \geqslant 2$, take the entourages $W = \bigcap_{j \in J} W_{f_j, 1}$ and $W' = \bigcap_{j \in J} W_{f_j, 1/2}$ of $(S, \mathcal{U}(\mathcal{T}))$ and the uniform covering $\mathcal{R} = \{W'[x]\}(x \in S)$, where

$$W'[x] = \{y \in S \,/\, (x, y) \in W'\}.$$

For any $x \in S$, $St(x, \mathcal{R}) \cap A_j \neq \varnothing$ for at most one index $j \in J$, since $W' \circ W' \subseteq W$.

If $(S, \mathbf{D})$ is a distal space, the relation δ given by $A \delta B \Leftrightarrow \{A, B\} \notin \mathbf{D}$ is an Efremovič proximity (see [1], [5]). Moreover:

PROPOSITION 7. - *The topology of (S, δ) is the $\mathbf{D}$-topology.*

PROPOSITION 8. - *Let (S, δ) be a proximity space and $\mathbf{D}(\delta)$ the collection of all families $\{X_i\}(i \in I)$ of subsets of S such that:*

(I) the set $I' = \{i \in I / X_i \neq \varnothing\}$ is finite;

(II) $X_{i_1} \delta^* X_{i_2}$ whenever $i_1 \neq i_2$ (3).

Then $\mathbf{D}(\delta)$ is a distal structure on S, and the $\mathbf{D}(\delta)$-topology is the topology of (S, δ).

Proof. $\mathbf{D}(\delta)$ satisfies the conditions of footnote (1).

We immediately obtain (1), (2), (3), (4), (5).

(6). Let $\mathcal{A} = \{A_i\}(i \in I) \in \mathbf{D}(\delta)$ and $I' = \{i \in I / A_i \neq \varnothing\}$ To construct $\mathcal{B}$, put $B_i = \varnothing$ for $i \in I - I'$. If $I' = \{i_0\}$, put $B_{i_0} = S$. If $\bar{\bar{I'}} \geq 2$ and $j \in I'$, for each $i \in I' - \{j\}$ take two subsets C_{ij} and C_{ji} of S such that $A_j \delta^* C_{ji}$, $A_i \delta^* C_{ij}$, $(S - C_{ij}) \delta^* (S - C_{ji})$; then put $B_j = \bigcap_{i \in I' - \{j\}} (S - C_{ji})$.

Finally, the $\mathbf{D}(\delta)$-topology is the topology of (S, δ), because, for any $x \in S$, a subset U of S is a $\mathbf{D}(\delta)$-nbd of x iff it is a neighbourhood of x in (S, δ).

REMARK 6. - Let $(S, \mathcal{T})$ be a T_{3a} topological space and δ a proximity on S compatible with $\mathcal{T}$. Then $\mathbf{D}(\mathcal{T}) \leq \mathbf{D}(\delta)$. In fact if $A \neq \varnothing$, $B \neq \varnothing$ and $A \delta^* B$, we find a continuous map $f : S \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = [1]$ (see [4], (3.15)).

REMARK 7. - Let (S, δ) be a proximity space. A distal structure $\mathbf{D}$ on S is said compatible with δ iff $\{A, B\} \in \mathbf{D} \Leftrightarrow A \delta^* B$. The distal structure $\mathbf{D}(\delta)$ from Proposition 8 is the coarsest distal structure

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(3) δ^* is the negation of δ.

on S compatible with δ. Therefore, if $\mathcal{U}$ is a uniformity on S inducing δ, we have $\mathbf{D}(\mathcal{U}) \leq \mathbf{D}(\delta)$.

PROPOSITION 9. - *Let (S, d) be a pseudometric space and $\mathbf{D}(d)$ the collection of all families $\{X_i\}(i \in I)$ of subsets of S such that:*

$$(\exists\, r > 0)\ (\forall i_1 \in I)\ (\forall i_2 \in I)\ (i_1 \neq i_2 \Rightarrow d(X_{i_1}, X_{i_2}) \geq r).$$

Then $\mathbf{D}(d)$ is a distal structure on S, and the $\mathbf{D}(d)$-topology is the pseudometric topology of (S, d).

Proof. We can easily see that conditions (D_1), (D_2), (D_3) hold. Finally the $\mathbf{D}(d)$-topology is the pseudometric topology of (S, d), because U is a $\mathbf{D}(d)$-nbd of x iff U contains $V(x, r) = \{y \in S / d(x, y) < r\}$ for some $r > 0$.

REMARK 8. - Let (S, d) be a pseudometric space, $\mathcal{T}$ its topology, $\mathcal{U}$ the uniformity of basis $\{\mathcal{R}_\varepsilon\}(\varepsilon > 0)$ where $\mathcal{R}_\varepsilon = \{V(x, \varepsilon)\}(x \in S)$, δ the proximity given by $A\,\delta\,B \Leftrightarrow d(A, B) = 0$. We easily obtain $\mathbf{D}(\mathcal{U}) = \mathbf{D}(d) \leq \mathbf{D}(\mathcal{T}) \leq \mathbf{D}(\delta)$. Finally the real line R with its standard metric is such that $\mathbf{D}(d) \neq \mathbf{D}(\mathcal{T})$. In fact, the family $\mathcal{F} = \{\{n\}\}(n \in N)$, where N denotes the set of all natural numbers, is such that $\mathcal{F} \in \mathbf{D}(\mathcal{T})$ and $\mathcal{F} \in \mathbf{D}(d)$.

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