A NOTE ON ELLIPTIC B.V.P. WITH JUMPING NONLINEARITIES (*)

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SOMMARIO. - *Viene studiato il problema al contorno con jumping nonlinearities su un dominio limitato regolare* $\Omega \subset \mathbf{R}^n$:

$$\begin{aligned} -\Delta u &= \lambda_+ u^+ - \lambda_- u^- + g(u) + h && \textit{in } \Omega \\ u &= 0 && \textit{su } \partial\Omega \end{aligned}$$

con g sublineare.

SUMMARY. - *We consider the boundary value problem with jumping nonlinearities on a bouded regular domain* $\Omega \subset \mathbf{R}^n$:

$$\begin{aligned} -\Delta u &= \lambda_+ u^+ - \lambda_- u^- + g(u) + h && \textit{in } \Omega \\ u &= 0 && \textit{on } \partial\Omega \end{aligned}$$

with g sublinear.

Introduction

The note treats the following b.v.p. with «jumping nonlinearities» on a bounded regular domain $\Omega \subset \mathbf{R}^n$:

$$\begin{aligned} -\Delta u &= \lambda_+ u^+ - \lambda_- u^- + g(u) + h && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{1}$$

with g sublinear in a sense which will be stated later on. This pro-

(*) Pervenuto in Redazione il 21 ottobre 1982.
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blem has been studied by many authors, see e.g. the references in this paper. We consider as in [4], [1], [5] h split into two parts

$$h = h_1 + t\,\varphi_k \tag{2}$$

where φ_k is an eigenfunction of $-\Delta$ corresponding to the k-th eigenvalue λ_k and take t as real parameter. In [4], [1], [5] k is such that $\lambda_k \in]\lambda_-, \lambda_+[$, while here we take $k = 1$ and suppose:

(I) $\qquad \lambda_1 < \lambda_- < \lambda_+ \qquad \lambda_\pm \neq \lambda_j$

(I) is assumed also in [5], asking moreover that $[\lambda_-, \lambda_+]$ contains exactly one eigenvalue of $-\Delta$ which is simple. Here we do not assume any condition of this kind. Fixed $h_1 \in L^2(\Omega)$, $h_1 \perp \varphi_1$, we take h as in (2) using t as a parameter and we will refer to the problem (1) as $(1)_t$. We study the problem reducing it to an asymptotically linear one (obtained substituting λ_+ with λ_-); this point of view has been used in [6] to prove a multiplicity result in the case: $\lambda_- < \lambda_1 < \lambda_+$. The paper is devided into 3 sections. In sect. 1 we state the notation and the results, in sect. 2 we evaluate the Leray-Schauder degree for some mappings, sect. 3 is devoted to the proofs of the theorems.

Section 1

Let E be the space $L^2(\Omega)$, where Ω is a given bounded open regular domain in $\mathbf{R}^n$, let g be a given function defined on $\Omega \times \mathbf{R}$ and valued on $\mathbf{R}$, which satisfies the Caratheodory condition i.e.

(g_1) $\quad g(x,t)$ is continuous in t for a.e. $x \in \Omega$ and is measurable in x for any $t \in \mathbf{R}$

and

(g_2) $\quad g$ is bounded

As in [6], (g_2) can be weakened asking

(g'_2) $\quad |g(x,t)| < c(a(x) + |u|^\alpha)$

where $\alpha \in \mathbf{R}, 0 \leq \alpha < 1, a \in E$. Here and throughout the following the letter c denotes a positive constant. We assume (g_2) only to simplify the computations, the idea of the proof is the same under (g'_2). E is an ordered Hilbert space whose norm will be denoted by $\|\cdot\|$ and whose scalar product by $(.,.)$; the positive cone P is, as usual, the set of the a.e. positive function. We set: $u^+ = \sup(u,o)$ and $u^- = (-u)^+$. We denote by g_* the Nemytskii operator induced by g and by K the resolvent operator $(-\Delta)^{-1}$. It is well-known that, by (g_{1-2}), g_* is a continuous bounded mapping on E and that K is a linear compact operator.

We will assume the non-resonance condition

(II) The problem:

$$\begin{aligned} \Delta u &= \lambda_+ u^+ - \lambda_- u^- && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{3}$$

has only the trivial solution.

Unfortunately we do not know a complete characterization, in terms of the eigenvalues λ_j of the pairs (λ_+, λ_-) such that (II) holds. However we suspect that (II) is a generic property and we can prove it in several cases (see e.g. [3], [5]) in which our results apply. We prove:

THEOREM 1: *Let* (I-II) *and* (g_{1-2}) *hold. Then at least one of the following two cases is true:*

(i) (1) *has at least a solution for any h.*

(ii) For any given h, there is a real number τ *such that* $(1)_t$ *has at least two solutions if* $\tau < |t|$.

THEOREM 2: *Let* (I-II) *and* (g_{1-2}) *hold, let also the total number of the eigenvalues* λ_j *(counted as many times as their multiplicity) in* $[\lambda_-, \lambda_+]$ *be odd. Then given h, there is a real number* τ *such that* $(1)_t$ *has at least two solutions, at least for all* $t < \tau$ *or for all* $t > \tau$.

Section 2

For $u \in E$, $s, t \in \mathbf{R}$, we set:

$$F_t(u) = u - K(\lambda_+ u^+ - \lambda_- u^- + g u + t \varphi_1) \tag{4}$$

$$F_t^{\pm}(s, u) = u - K(\lambda_\pm u + s g_*(u + \frac{t}{\lambda_1 - \lambda_\pm} \varphi_1)) \tag{5}$$

LEMMA 6: *There exist* $r, \varepsilon \in \mathbf{R}_+$ *such that if* $u \in E$ *verifies:*

$$\| F_t(s, u) \| < \varepsilon \| u \| \tag{7}$$

for some $s, t \in \mathbf{R}$, $0 \le s \le 1$, *it then:* $\| u \| < r$.

Proof: Using assumption (I) we take ε such that $\forall\, u \in E$:

$$\| u - \lambda_\pm K u \| > 2 \varepsilon u$$

From (7) it follows: $\| g_*(u + \frac{t}{\lambda_1 - \lambda_\pm} \varphi_1) \| \geq \varepsilon \| u \|$ and therefore since g_* is bounded we prove the statement. ∎

We fix r as in the preceding lemma and set

$$i_t^{\pm} = \deg (B(0, r), F_t^{\pm}(1, .), 0)$$

Lemma 6 implies that $i_t^{\pm}$ is well defined. Let $k_\pm$ be the positive integer such that:

$$\lambda_{k_\pm} < \lambda_\pm < \lambda_{k_\pm + 1}$$

LEMMA 8: $\forall t \in \mathbf{R} : i_t^\pm = (-1)^{k_\pm}$

Proof: We must only remark that:

$$i_t^\pm = \deg\,(B(0,r), F_t^\pm\,(0,.), 0).$$

This is true since lemma 6 insures the admissibility of the homotopy $F_t^\pm$. ■

We set: $B_t^\pm = B\,(\frac{t}{\lambda_1 - \lambda_\pm}\varphi_1, r)$ and denote by V_n the subspace of E spanned by the eigenfunctions of $-\Delta$ related to the first n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. From the strong positiveness of φ_1 we get, for any n two real numbers $a_n^\pm$ such that

$$B_t^+ \cap V_n \subset P \text{ if } t < a_n^+ \text{ and } B_t^- \cap V_n \subset -P \text{ if } a_n^- < t.$$

LEMMA 9: *There exist a positive integer n such that*

$$\deg\,(B_t^\pm, F_t, 0) = i_t^\pm \tag{10}$$

if $t < a_n^+$ *in the* $+$ *case and if* $a_n^- < t$ *in the* $-$ *case.*

Proof. We only prove the $+$ case. Let us define the homotopy:

$$\eta_t\,(s, u) = F_t(u) + s(\lambda_+ - \lambda_-)\,Ku^-$$

it is easy to see that

$$\eta_t\,(1, u) = F_t^+\,(1, u - \frac{t}{\lambda_1 - \lambda_+}\varphi_1) \tag{11}$$

therefore we have by translation:

$$\deg\,(B_t, \eta_t(1,.), 0) = i_t^\pm.$$

We have only to prove that if $t < a_n^+$ then η_t is admissible. From lemma 6 and (11), we have $\forall t \in \mathbf{R}$, $\forall\, u \in \partial B_t$:

$$\|\,\eta_t\,(1, u)\,\| \geqq \varepsilon\,\|\,u - \frac{t}{\lambda_1 - \lambda_+}\varphi_1\,\| \tag{12}$$

Suppose $\eta_t(s, u) = 0$ for some $s \in [0, 1]$ and some $u \in \partial B_t$.

Write $u = v_n + w_n$, with $v_n \in V_n$ and $w_n \perp V_n$. From (12) we have:

$$\|\,u^-\,\| \geq \frac{\varepsilon}{(\lambda_+ - \lambda_-)\,\|\,K\,\|}\|\,u - \frac{t}{\lambda_1 - \lambda_+}\varphi_1\,\| = \frac{\varepsilon\, r}{(\lambda_+ - \lambda_-)\,\|\,K\,\|} \tag{13}$$

while, for $t < a_n^+$, $v_n \in B_t^+ \cap V_n \subset P$.

Therefore

$$\|\,u^-\,\| = \|\,(v_n + w_n)^-\,\| \leq \|\,w_n^-\,\| \leq \|\,w_n\,\|$$

and by (13):

$$\|\,w_n\,\| \geq \frac{\varepsilon\, r}{(\lambda_+ - \lambda_-)\,\|\,K\,\|}$$

$$\| -\Delta u - \lambda_+ u - t\varphi_1 \| \geq \| \Delta w_n - \lambda_+ w_n \| \geq \frac{(\lambda_{n+1} - \lambda_+)\, \varepsilon r}{(\lambda_+ - \lambda_-)\, \| K \|} \tag{14}$$

The assumption $\eta_t(s, u) = 0$ is equivalent to

$$-\Delta u - \lambda_+ u - t\varphi_1 = (s-1)(\lambda_+ - \lambda_-)\, u^- + g u \tag{15}$$

Using the above estimates one proves that the right-hand side of (15) is bounded and gets in contraddiction with (14) taking n large enough. The —case is treated in a similar way. ■

Section 3

We state without proof, which is not difficult and uses a standard degree argument, the following

LEMMA 16: *If* (1) *has no solution for some* $h \in E$, *then* $\exists\, \bar{R} \in \mathbf{R}$ *such that* $\forall\, R \geq \bar{R}$:

$$\deg(B(0, R), F_t, 0) = 0 \tag{17}$$

Proof of Theorem 1 - Suppose that (1) has no solution for some h. Fix n as in lemma 9 and take $\tau = \max$

ourselves to the case $t < 0$ i.e. $t < a_n^+$. Take $R > \bar{R}$ such that $B_t^+ \subset B(0, R)$. We suppose $h_1 = 0$, which is not restrictive since h_1 can be incorporated in g. (1) is equivalent to the equation $F_t = 0$, and therefore (10) and (17) give the existence of at least two solutions. ■

Proof of Theorem 2 - Take τ as before and suppose by contraddiction that there exist $t_+ < -\tau$ and $t_- > \tau$ such that $(1)_{t_+}$ and $(1)_{t_-}$ have at most one solution. As before, we can suppose $h_1 = 0$ and by the excision property of the topological degree we get:

$$\deg(B(0, R), F_{t_\pm}, 0) = \deg(B_t^\pm, F_{t_\pm}, 0) = (-1)^{k_\pm}$$

if $B_{t_\pm} \subset B(0, R)$. Since $k_+ - k_-$ is odd, F_{t+} and F_{t-} cannot be homotopic on $B(0, R)$. Therefore we get the existence of an unbounded set Σ of pairs (t, u) such that $F_t(u) = 0$ and $t \in [t_+, t_-]$. Take $(t_n, u_n) \in \Sigma$ with $\lim_n \| u_n \| = +\infty$. $F_{t_n}(u_n) = 0$ is equivalent to

$$\begin{aligned} -\Delta u_n &= \lambda_+ u_n^+ - \lambda_- u_n^- + g(u_n) + t_n \varphi_1 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{18}$$

and setting $v_n = \| u_n \|^{-1} u_n$, by (18) we can suppose that $(v_n)_{n \in N}$ has a convergent subsequence to a limit $v \neq 0$. Taking the limit in (18) we have that v is nontrivial solution of (3) and we get a contraddiction to the assumption (II). ■

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