BASES OF CONVERGENCE AND DIAGONAL CONDITIONS (*)

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SOMMARIO. - *Per le strutture di convergenza (di successioni ordinarie) si confrontano quattro condizioni di tipo diagonale; si costruisce un esempio di convergenza che, aggiunto alle implicazioni già note, dimostra che le quattro condizioni sono a due a due non equivalenti.*

SUMMARY. - *For convergences (of ordinary sequences) four diagonal conditions are considered; a counterexemple is given allowing to conclude, together with some previous results, that any two among the four conditions are not equivalent.*

1. When studying convergences, we usually assume the following conditions:

F Each subsequence of a convergent sequence is convergent to the same limit.

U If from each subsequence of x_n we can select a subsequence which is convergent to x, then x_n is convergent itself to x.

S A constant sequence $x, x, \ldots$ is convergent to x.

In minute investigations these three conditions are not sufficient. The following condition turned out to be very useful and was considered by many authors independently.

D If $x_{mn} \to x_m$ for each $m \in N$ and $x_m \to x$, then there exists a sequence of indices $p_n \to \infty$ such that $x_{np_n} \to x$.

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By N we denote the set of all positive integers. Condition **D** says that if we have an infinite matrix whose n-th row is convergent to x_n and the sequence x_n converges to x, then the matrix has a diagonal convergent to x. By a *diagonal* we mean any sequence whose n-th element is in the n-th row of the matrix and which has at most a finite number of elements in each column. A subsequence of a diagonal is called *subdiagonal.*

Condition **D** is not the only condition of diagonal type appearing in literature. In [1] there are considered three other diagonal conditions. One of them ensures existence of a subdiagonal instead of the diagonal.

D' If $x_{mn} \to x_m$ for each $m \in N$ and $x_m \to x$, then there exist two sequences of indices $p_n\ q_n \to \infty$ such that $x_{p_n q_n} \to x$.

In another condition, limits of rows are assumed to be equal:

D₀ If $x_{mn} \to x$ for each $m \in N$, then there exists a sequence of indices $p_n \to \infty$ such that $x_{np_n} \to x$.

It is possible to join assumptions of $\mathbf{D}_0$ with thesis of **D'**:

$\mathbf{D'}_0$ If $x_{mn} \to x$ for each $m \in N$, then there exist two sequences of indices $p_n, q_n \to \infty$ such that $x_{p_n q_n} \to x$.

We have the following trivial interferences:

$$\begin{array}{ccc} \mathbf{D} & \longrightarrow & \mathbf{D}_0 \\ \downarrow & & \downarrow \\ \mathbf{D'} & \longrightarrow & \mathbf{D'}_0 \end{array}$$

It is not difficult to give an example of convergence satisfying condition $\mathbf{D}_0$ but not **D'** (see [2], p. 87). The aim of this note is to give, under Continuum Hypothesis, an example of convergence **D'** but not $\mathbf{D}_0$. In this way we proved that if we assume Continuum Hypothesis, then any two conditions from among $\mathbf{D}, \mathbf{D'}, \mathbf{D}_0, \mathbf{D'}_0$ are not equivalent.

To construct this example we use the notion of a base of convergence at a point introduced by Dolcher in [2]. At the end of this note we discuss some facts concerning diagonal conditions and bases of convergences.

2. Let X be a nonempty set endowed with a convergence satisfying conditions **F U S**. To define a base of convergence we shall use the notation introduced in [4] and [5]. If $\mathfrak{A}$ is any family of sequences of elements from X, then by $S\mathfrak{A}$ we denote the family

of all subsequences of sequences from $\mathfrak{A}$. In symbols:

$$S\mathfrak{A} = \{\langle x_n \rangle \in X^N;\ \exists \langle y_n \rangle \in \mathfrak{A} \quad x_n < y_n\},$$

where $x_n < y_n$ means that x_n is a subsequence of y_n. By $E\mathfrak{A}$ we denote the family of all sequences from X^N which have a subsequence in $\mathfrak{A}$. In symbols:

$$E\mathfrak{A} = \{\langle x_n \rangle \in X^N;\ \exists y_n < x_n \quad \langle y_n \rangle \in \mathfrak{A}\}.$$

The family of all sequences whose all subsequences belong to $\mathfrak{A}$ is denoted by $R\mathfrak{A}$. In symbols:

$$R\mathfrak{A} = \{\langle x_n \rangle \in X^N;\ \forall y_n < x_n \quad \langle y_n \rangle \in \mathfrak{A}\}.$$

By $RES\mathfrak{A}$ we mean a composition of operations S, E, R on $\mathfrak{A}$.

A family of sequences $\mathfrak{B}_x$ is said to be a *base of convergence* at x, iff $RES\mathfrak{B}_x$ is the family of all sequences converging to x.

We can say that $\mathfrak{B}_x$ is a base at x if each sequence converging to x has a common subsequence with a sequence from $\mathfrak{B}_x$. A convergence generated by such bases at each point of X satisfies conditions **F U**. If we additionally assume that for each $x \in X$ the constant sequence $x, x, \ldots$ belongs to $\mathfrak{B}_x$, then we obtain a convergence satisfying condition **S**. It is obvious that every convergence having properties **F U** has a base at each point. It suffices to take for $\mathfrak{B}_x$ the family of all sequences converging to x. In applications, bases of minimal number of sequences are of main importance.

Two sequences will be called *independent*, if they have no common subsequence. So, x_n and y_n are independent iff $S\{\langle x_n \rangle\} \cap S\{\langle y_n \rangle\} = \phi$. Saying a *family of independent sequences*, we mean that any two sequences from it are independent. If a sequence x_n is independent of each sequence from a family of sequences $\mathfrak{F}$, we say that x_n is independent of $\mathfrak{F}$.

Now we introduce some lemmas which are useful in presentation of the example to be constructed.

LEMMA 1. *Let $\mathfrak{F}$ be a family of independent sequences. If $\mathfrak{F}_1, \mathfrak{F}_2 \subset \mathfrak{F}$ and $\mathfrak{F}_1 \cap \mathfrak{F}_2 = \phi$, then $RES\,\mathfrak{F}_1 \cap RES\,\mathfrak{F}_2 = \phi$.*

Proof. Assume, on the contrary, that $\langle x_n \rangle \in RES\,\mathfrak{F}_1 \cap RES\,\mathfrak{F}_2$. Then there exists a subsequence y_n of x_n which is a subsequence of a sequence from $\mathfrak{F}_1$. Farthermore, there exists a subsequence z_n of y_n which is a subsequence of a sequence from $\mathfrak{F}_2$. Hence, z_n is a subsequence of two different sequences from $\mathfrak{F}$, but this is in contradiction with independency of sequences from $\mathfrak{F}$.

LEMMA 2. *Let, for each $x \in X$, $\mathfrak{B}_x$ be a base of convergence at x. Condition* **D'** *is equivalent to the following condition*

* *If, for each $m \in N$, $\langle x_{mn} \rangle \in S\,\mathfrak{B}_{x_m}$ and $\langle x_m \rangle \in S\,\mathfrak{B}_x$, then there*

exist two sequences of indices $p_n, q_n \to \infty$ *such that* $x_{p_n q_n} \to x$.

Proof. From each row of any matrix with convergent rows we can select a subsequence which is a subsequence of a sequence from the base.

This lemma allows us, when verifying **D′**, to restrict our considerations to matrices whose rows are subsequences of sequences from the base. Similar lemmas can be formulated for conditions **D**, $\mathbf{D}_0$, $\mathbf{D}'_0$.

We say that a matrix M_0 is a *submatrix* of a matrix M, iff the n-th row of the matrix M_0 is a subsequence of the n-th row of the matrix M.

LEMMA 3. *Conditions* $\mathbf{D}_0$ *is equivalent to the following condition*

** *If each row of a matrix M converges to x, then there exists a submatrix of M such that all its diagonals converge to x.*

Proof. Let each row of a matrix M converge to x and let M_1 be a matrix whose rows are from M and each row repeats in M_1 infinitely many times. For M_1 we can take the following matrix

$$\begin{matrix} x_{11} & x_{12} & x_{13} & \dots \\ x_{21} & x_{22} & x_{23} & \dots \\ x_{11} & x_{12} & x_{13} & \dots \\ x_{21} & x_{22} & x_{23} & \dots \\ x_{31} & x_{32} & x_{33} & \dots \\ x_{11} & x_{12} & x_{13} & \dots \\ \dots & \dots & \dots & \dots \end{matrix}$$

where x_{mn} are the elements of M. Since each row of M_1 is convergent to x, there exists a diagonal of M_1 convergent to x, by $\mathbf{D}_0$. This diagonal has a common subsequence with each row of M. The matrix of this subsequences has the required properties. Thus ** follows from $\mathbf{D}_0$.

The converse implication is obvious.

LEMMA 4. *Let* $\mathfrak{F}$ *be a family of independent sequences and let M be a matrix the rows of which are subsequences of different sequences from* $\mathfrak{F}$. *Let* $\mathfrak{G} = \{g_1, g_2, \dots\}$ *be a countable subset of* $\mathfrak{F}$. *If each row of M has infinitely many different elements, then there exists a subdiagonal of M independent of* $\mathfrak{G}$.

Proof. We divide the proof into two cases.

First case. There are infinitely many rows in M not belonging to $S\,\mathfrak{G}$. Denote by M^* the matrix containing only these rows. Then from the n-th row of M^* we take an element which does not appear in $g_1, \dots, g_n$. In this way we obtain the sequence which is a subdiagonal of M independent of $\mathfrak{G}$.

Second case. Almost all rows are sequences from $S\mathfrak{S}$. These sequences form a matrix M^*. Let $e_1, e_2, \dots$ be the remaining sequences of $\mathfrak{S}$, i.e., sequences from $\mathfrak{S}$ which are independent of rows of M^*. Then from the n-th row $(n \geqslant 2)$ we take an element which does not appear in the initial $n-1$ rows of M^* or in $e_1, \dots, e_n$. In this way we obtain a sequence which is a subdiagonal of M, independent of $\mathfrak{S}$.

3. To construct the required example we shall use the following

THEOREM 1. *Let K be a countable set. Then there exists a family $\mathfrak{F}$ of infinite subsets of K such that*

i) *$\mathfrak{F}$ has the power of Continuum,*

ii) *If $F_1, F_2 \in \mathfrak{F}$ and $F_1 \neq F_2$, then $F_1 \cap F_2$ is a finite set.*

iii) *For each countable subset G of K there exists $F \in \mathfrak{F}$ such that $F \cap G$ is an infinite set.*

A simple proof of this theorem is given in [1]. (The Kuratowski-Zorn Lemma is used).

EXAMPLE. Let $X = \{\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots\}$. By Theorem 1, there exists a family $\mathfrak{F}$ of subsequences of $\varepsilon_1, \varepsilon_2, \dots$ such that

I $\mathfrak{F}$ has the power of Continuum,

II If $a, b \in \mathfrak{F}$ and $a \neq b$, then a and b have no common subsequence,

III Each subsequence of $\varepsilon_1, \varepsilon_2, \dots$ has a subsequence belonging to $S\mathfrak{F}$.

Assuming Continuum Hypothesis, we arrange all matrices whose rows are subsequences of different sequences from $\mathfrak{F}$ in a transfinite sequence M_α, where $\alpha < \omega_1$. By ω_1 we denote the first uncountable ordinal number. Since $\mathfrak{F}$ is a family of independent sequences, each row of M_α is a subsequence of only one sequence from $\mathfrak{F}$. Consequently, the following set

$\Phi M_\alpha = \{a \in \mathfrak{F};$ such that there exists a row in M_α which is a subsequence of $a\}$

is countable for each $\alpha < \omega_1$.

By transfinite induction, we shall define two subsets $\mathfrak{A}$ and $\mathfrak{B}$ of the family $\mathfrak{F}$. Let

$$A_1 = \{\Phi M_1 \cup \{a_1\}\}$$

where a_1 is an element of $\mathfrak{F}$ containing a subsequence which is a subdiagonal of M_1. Such an element exists, by III. Let

$$B_1 = \{b_1\}$$

where b_1 is an element of $\mathfrak{F}$ containing a subsequence which is a subdiagonal of M_1 and is independent of A_1. Such an element exists, by III and Lemma 4.

Assume that we already defined A_α and B_α for all $\alpha < \beta$. We define A_β and B_β by the equalities:

$$A_\beta = \begin{cases} \bigcup_{\alpha<\beta} A_\alpha & \text{if} \quad \Phi M_\beta \cap \bigcup_{\alpha<\beta} B_\alpha \neq \phi \\ \bigcup_{\alpha<\beta} A_\alpha \cup \Phi M_\beta \cup \{a_\beta\} & \text{if} \quad \Phi M_\beta \cap \bigcup_{\alpha<\beta} B_\alpha = \phi \end{cases}$$

where a_β is an element of $\mathfrak{F}$ independent of $\bigcup_{\alpha<\beta} B_\alpha$ and containing a subdiagonal of M_β (such an element exists, by III and Lemma 4; $\bigcup_{\alpha<\beta} B_\alpha$ is a countable subset of $\mathfrak{F}$) and

$$B_\beta = \bigcup_{\alpha<\beta} B_\alpha \cup \{b_\beta\}$$

where b_β is an element of $\mathfrak{F}$ independent of A_β containing a subdiagonal of M_β.

Let

$$\mathfrak{A} = \bigcup_{\alpha<\omega_1} A_\alpha \quad \text{and} \quad \mathfrak{B} = \bigcup_{\alpha<\omega_1} B_\alpha$$

Consider convergence in X which at ε_0 has a base $\{\langle \varepsilon_0, \varepsilon_0, \ldots \rangle, \mathfrak{A}\}$ and at points ε_n a base $\{\langle \varepsilon_n, \varepsilon_n, \ldots \rangle\}$.

The convergence satisfies condition **D'**. In fact, let M be any matrix satisfying assumptions in *. If there are in M infinitely many rows which are constant sequences, then obviously M has a convergent subdiagonal. Similarly, it is not difficult to point out a convergent subdiagonal in the case when there are infinitely many rows which are subsequences of one sequence from $\mathfrak{A}$. In other cases, after canceling some rows, we obtain one of matrices M_α which have convergent subdiagonal.

The convergence does not satisfy $\mathbf{D}_0$. By Lemma 1, no sequence from $S\mathfrak{B}$ is convergent. Thus, each matrix whose n-th row is a subsequence of the n-th row of M_1 has a nonconvergent subdiagonal. Consequently, by Lemma 3, the convergence cannot satisfy $\mathbf{D}_0$.

Note that, in the above example, each convergent sequence converges to a unique limit.

4. In this section we prove some facts concerning the bases of convergence and diagonal conditions.

We say that a convergence satisfies condition $\mathbf{D}_0$ for a point x, if each matrix whose rows converge to x has a diagonal convergent to x.

THEOREM 2. *If a convergence has a countable base at x and satisfies condition $\mathbf{D}_0$ for x, then it has a finite base at x.*

Proof. Let $b_1, b_2, \ldots$ be a base at x. We put $a_1 = b_1$. Then we denote by a_2 the sequence of all elements which do not appear in a_1 from the first sequence b_n in which there are infinitely many such elements. Generally, we denote by a_n the sequence of all elements which do not appear in $a_1, \ldots, a_{n-1}$ from the first sequence b_m in which there are infinitely many such elements. If for some n it is impossible, then $a_1, \ldots, a_{n-1}$ is a base at x. In other case we obtain the sequence $a_1, a_2, \ldots$ which is a countable base of independent sequences. Then no diagonal of the matrix which rows are $a_1, a_2, \ldots$ is convergent. This contradicts $\mathbf{D}_0$.

REMARK 1. If a convergence satisfies condition $\mathbf{D}_0$ for x, then there does not exist a countable base at x of independent sequences.

REMARK 2. If a convergence has a finite base at x, then satisfies $\mathbf{D}_0$ for x.

From Lemma 3 we easily obtain

THEOREM 3. *Let, for $n = 1, 2, \ldots, G_n$ be convergences on X_n and let G be the product convergence on $X = X_1 \times X_2 \times \ldots$ If G_n satisfy $\mathbf{D}_0$, then G satisfies $\mathbf{D}_0$.*

Proof. Let M be any matrix of elements from X satisfying assumptions in $\mathbf{D}_0$. Then there exists a submatrix M_1 of M whose all diagonals are convergent on X_1. Next we select a submatrix M_2 of M_1 whose all diagonals are convergent on X_2, and so on. We obtain a sequence M_n of submatrices of M. Let M^* be a matrix whose n-th row is equal to the n-th row of M_n. The matrix M^* has all its diagonals G-convergent, so the convergence G satisfies condition $\mathbf{D}_0$.

REMARK 3. From Lemma 3 it follows that, in condition $\mathbf{D}_0$, we can assume that p_n is increasing.

For more results on diagonal conditions see [3] and [6].

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