ON COMPLEX STRICTLY CONVEX SPACES III (*)

by VASILE I. ISTRĂŢESCU (in Frankfurt am Main) (**)

SOMMARIO. - *Si danno alcuni risultati su una geometria di spazi di Banach in connessione con alcuni problemi ad essa relativi.*

SUMMARY. - *We give some results related to a geometry of Banach spaces in connection with some related problems.*

0. Introduction.

In the paper [8] the class of Banach spaces introduced by Thorp and Whitley are studied in connection with the Köthe's problem which in its general form is as follows: To what extent is a property of a Banach space inherited by its quotient spaces? In the original form this was about the smoothness and rotundity.

In what follows we give further results about the class of spaces introduced by Thorp and Whitley which are called « complex strictly convex spaces». First we show, by an example, that the Fujiwara-Minkowski theorem fails to hold for the case of complex extreme points. Next we give a result about the form of the set of complex extreme points of the unit ball of a complex Banach space. Also we give some results about the interpolation spaces and complex strictly convex spaces. Some aspects of the complex extremal structure in the space of operators are treated in the final part.

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(**) Indirizzo dell'Autore: Fachbereich Mathematik der Johann Wolfgang Goethe-Universität, Robert Mayer Strasse 6-10, 6000 Frankfurt am Main 1 (Germania federale).

1. Complex extreme points.

Let X be a complex Banach space and $x \in X$ with $\|x\| = 1$.

DEFINITION 1.1. The point x is called ccmplex extreme point if $\|x+\xi y\| \leq 1$ for all $|\xi| \leq 1$ implies $y=0$.

DEFINITION 1.2. A complex Banach space is called « complex strictly convex » iff all x, $\|x\| = 1$ are complex extreme points.

REMARK 1.3. In the Definition 1.1. we can consider other bounded closed convex sets and the definition of the complex extreme points is as follows: the point $x \in C$ is a complex extreme point of C if

$$x+\xi y \in C \text{ for all } \xi,\ |\xi| \leq 1$$

then $y=0$.

2. On the complex extreme points for the sum of sets.

Let M_1 and M_2 be two sets in the complex Banach space X. Then the sum M_1+M_2 is defined as the set

$$M_1+M_2=\{m_1+m_2,\ m_1 \in M_1 \text{ and } m_2 \in M_2\}.$$

As is well known if M_i are compact convex sets in a Banach space then for any extreme point of M_1+M_2 there exists $m_1 \in M_1$ and $m_2 \in M_2$, uniquelly determined extreme points such that

$$m=m_1+m_2.$$

This is the so called Minkowski-Fujiwara theorem.

We show now that a similar assertion for the case of complex extreme points is not true.

EXAMPLE 2.1. Let $X=L([0,1], B^1, dx)$ and $M_1=M_2=\{x, \|x\| \leq 1\}$. In this case M_1+M_2 has a complex extreme point the function

$$f(x)=1+2x$$

and we show that there exists f_1, f_2, g_1 and g_2 complex extreme points of M_1 such that

$$f(x)=f_1(x)+f_2(x)=g_1(x)+g_2(x).$$

Indeed we can take

$$f_1(x)=1/2+x=f_2(x)$$

$$g_1(x)=1, \quad g_2(x)=2x$$

and it is easy to see that these satisfy our assertion.

3. On the set of complex extreme points of a Banach space.

If X is a complex Banach space we are interested in the form of the set $\mathrm{Ext}_c S_1$, the set of all complex extreme points of

$$S_1=\{x, \|x\| \leq 1\}.$$

THEOREM 3.1. The set $\mathrm{Ext}_c S_1$ is G_δ-set.

PROOF. Our proof is valid in the following more general setting: C is a metrizable convex set in a complex topological space.

Indeed, if we suppose that the topology is given by the metric d then for each integer n we define the set

$$F_n=\{y \in C, x=y+\xi z \in C \text{ and } z \in X, |\xi| \leq 1, d(y, z) \geq 1/n\}.$$

From the definition it is clear that F_n are closed and a point of C is not a complex extreme point iff is not in some F_n. Thus the complement of the complex extreme point is an F_σ.

In connection with this result as well as with the Krein-Milman theorem we mention the following problem: suppose that the complex Banach space X has the property that each bounded closed convex subset is the convex hull of its complex extreme points. It follows that Reinwater's theorem holds in this case?

4. Complex strict covexity and interpolation spaces.

In a recent paper B. Beauzamy has proved a number of interesting results about the geometry of interpolation spaces introduced by Lions and Peetre [3]. In what follows we give a result about strict complex spaces and interpolation spaces.

For the reader's convenience as well as for the future use we recall here the norms which we use.

Suppose that A_0 and A_1 are two Banach spaces contained in a locally convex space A and we suppose that the embedding is continuous. We consider the space of sequences $u=(u_n)$ such that

$$\{e^{\xi_0} u_n\} \in l^{p_0}(A_0)$$

$$\{e^{\xi_1} u_n\} \in l^{p_1}(A_1)$$

and denote this set as $w(p_0, \xi_0, A_0;\ p_1, \xi_1, A_1)$.

The following norm are to be used in what follows:

$$||u|| = [\,\Sigma \inf_{u_0+u_1=u} \max(||e^{\xi_0 n} u_0||^p_{A_0},\ ||e^{\xi_1 n}\, u_1||^p_{A_1})^{1/p}].$$

and using the gauge of the convex set

$$e^{-\xi_0 n} B_0 + e^{-\xi_1 n} B_1$$

the above norm can be writen in the form,

$$||u|| = (\Sigma\, ||u||_n{}^p)^{1/p}$$

where $||\ ,\ ||_n$ is the gauge, and B_0 respectively B_1 are the unit balls of A_0 respectively A_1.

It is worth to mention the following relation

$$e^{-\xi_0} B_0 + z\, e^{-\xi_1} B_1 = e^{-\xi_0} B_0 + e^{-\xi_1} B_1$$

which is true for all $z = 1, -1, i, -i$.

We suppose that there exists an injective application of A_0 in A_1, i: $A_0 \rightarrow A_1$ with the property that $i(B_0)$ is weakly compact in A_1. In this case there exists for each u the elements u_0 and u_1 with the following properties:

1. $u = u_0 + u_1,\ u_0 \in A_0,\ u_1 \in A_1,$

2. $\max(||\, e^{\xi_0 n} u_0||_{A_0},\ ||e^{\xi_1 n} u_1||_{A_1}) =$

$$= \inf_{\tilde{u} + \tilde{u}_1 = u} \max(||e^{\xi_0 n} \tilde{u}_0||_{A_0},\ ||e^{\xi_1 n}\, u_1||_{A_1})$$

for each integer n.

Our result is the following,

THEOREM 4.1. If A_0 and A_1 are as above and A_0 is dense in A_1 then $(A_0, A_1)_{\vartheta,p}$ is a complex strictly convex space when A_0 or A_1 is a complex strictly convex space.

PROOF. Let u, $||u|| = 1$ and suppose that there exists v such that for all z, $|z| \leq 1$, $||u+zv|| \leq 1$. As is easy to see we can consider only the values $z=1, -1, i, -i$.

Let $u_0(n), u_1(n), v_0(n)$ and $v_1(n)$ be the elements having the properties stated in 1 and 2.

Since A_0 is dense in A_1 we can choose $u_0(n)$ and $u_1(n)$ such that

$$||e^{\xi_0 n} u_0(n)||_{A_0} = ||e^{\xi_1 n} u_1(n)||_{A_1},$$

and similarly for v.

Suppose that A_0 is a complex strictly convex space. From the form of $||\ ,||_n$ it follows that we find for each n, the elements $a_n, \tilde{a_n}, b_n$ and $\tilde{b_n}$ such that

$$u+v=(a_n+b_n)\,k_n$$

$$u-v=(\tilde{a_n}+\tilde{b_n})\,k_n$$

and

$$\{a_n, \tilde{a_n}\}\subset e^{\xi_0 n} B_0\,, \qquad \{b_n, \tilde{b_n}\}\subset e^{\xi_1 n} B_1\,.$$

From these it is clear that

$$u=1/2\,\{a_n+\tilde{a_n}+b_n+\tilde{b_n}\}\,k_n$$

$$v=1/2\,\{(a_n-\tilde{a_n})+(b_n-\tilde{b_n})\}\,k_n$$

and thus

$$u+zv=k_n\{1/2\,(a_n+\tilde{a_n})+1/2\,(b_n+\tilde{b_n})+$$
$$+z/2\,(a_n-\tilde{a_n})+z/2\,(b_n-\tilde{b_n})\}=$$
$$=k_n\{1/2\,(a_n+\tilde{a_n})+z/2\,(a_n-\tilde{a_n})+1/2\,(b_n+\tilde{b_n})+z/2\,(b_n-\tilde{b_n})\}$$

which gives that for all $z=1, -1, i, -i$

$$||u+zv||_n=k_n$$

and thus

$$1=\Sigma \max\{||e^{\xi_{0}n}(u_0(n)+z\, v_0(n))/2||^p_{A_0},\ ||e^{\xi_1 n}(u_1(n)+zv_1(n))/2||^p_{A_1}\}.$$

This gives further that

$$1=\Sigma\, ||e^{\xi_0 n}(u_0(n)+zv_0(n))/2||^p_{A_0}=\Sigma\, ||e^{\xi_1 n}(u_1(n)+zv_1(n))/2||^p_{A_1}$$

and this implies easy that $v=0$.

In a similar way we can prove the assertion if A_1 is complex strictly convex space.

REMARK 4.2. It appears of interest to know if the assertion of the theorem is true for other norms.

REMARK 4.3. It is possible that $(A_0, A_1)_{\vartheta,p}$ to be complex strictly convex if A_0 and A_1 are not both complex strictly convex spaces. In the case of uniformly convex spaces there exists an example: L^2 which interpolates between L^1 and L^∞.

REMARK 4.4. It is possible to give a more general treatement using an appropriate generalization of uniformly convexifiant operators of Beauzamy? .

5. **Complex extremal structure in operator spaces.**

In connection with the existence of the lifting, A. and C. T. Ionescu Tulcea proved an interesting generalization of the Arens-Kelley characterization of the extreme points of $(C(K))^*$ where K is a compact Hausdorff space. For the extensions as well as for further references of Tulcea's result we refer to [5].

In the paper of Morris and Phelps an interesting class of operators was considered, namely the class of « nice operators » which are defined as follows: an operator $T: E \to F$, E, F Banach spaces is called « nice » if

$$T^*(\text{Ext}\ S(F^*)) \subset \text{Ext}\,(E^*)$$

where $S(\ )$ denotes the unit ball and Ext ( ) denotes the set of extreme points. In what follows we consider a class of operators suggested by the class of Morris and Phelps and the complex extreme points.

If X is a complex Banach space and M is a convex set in X then $\mathrm{Ext}_c M$ denotes the set of all complex extreme points of M.

DEFINITION 5.1. An operator $T: E \to F$, E and F complex Banach spaces, is called «complex nice» if

$$T(\mathrm{Ext}_c S(E)) \subset Ext_c(S(E)).$$

DEFINITION 5.2. An operator $T \in L(E, F)$ is called complex extreme point if

$$T \in \mathrm{Ext}_c(S(L(E, F))).$$

The following result gives a first relation between these classes of operators.

THEOREM 5.3. If T is a complex nice operator then T is a complex extreme operator if E is a complex strictly convex space.

PROOF. Suppose that T is not a complex extreme point. In this case we find an operator $S \in L(E, F)$ such that for all z, $|z| \leq 1$

$$\|T + z S\| \leq 1.$$

Let x be any complex extreme point of $S(E)$ and we show that $Sx = 0$. We can suppose without loss of generality that $\|x\| = 1$.

We have further,

$$\|Tx + z Sx\| \leq 1$$

and since Tx is a complex extreme point we obtain that $Sx = 0$.

Since E is a complex strictly convex space the assertion is proved.

REMARK 5.4. The theorem is valid under the weaker condition: the unit ball is the closed convex hull of its complex extreme points.

THEOREM 5.5. If E is a complex Banach space, Y a compact Hausdorff space and T be a complex extreme operator of $L(E, F)$ with $F = C(Y)$ then the set

$$\{y \in Y, \|T^*(y)\| = 1\}$$

is dense in Y.

PROOF. Since the map

$$y \to \|T^* (y)\|$$

is lower semi-continuous, the sets

$$G_n = \{y \in Y, \|T^* (y)\| > 1 - 1/n\}$$

are open subsets. We show now that for each n, G_n is dense in Y.

Let $f \in C(Y)$ such that $0 < f < 1$ and $f/G_n = 0$. If $\mu \in S(E^*)$ is any nonzero functional we define the map $F^*: Y \to E^*$ by the relation

$$F^* (y) = 1/n f(y) \mu$$

and clear we have

$$\|T^* (y) + z F^* (y)\| \leq 1$$

for all complex numbers z, $|z| \leq 1$. Since T is complex extreme operator we obtain that $f = 0$. From Baire's theorem the assertion follows.

This result can be generalized to the following setting: E is an L^1 space, F an arbitrary space and T a complex extreme operator. The assertion is as follows.

THEOREM 5.6. If we identify E^* with a space of the form $C(Y)$ for some Y, then

$$\{y, y \in Y, \|T^* y\| = 1\}$$

is dense in Y.

This result is very useful in the following problem: to characterize the complex extreme operators between L^1 spaces.

For results in this direction as well as for related results see [10].

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