A GLOBAL VERSION OF A LINEAR GOURSAT PROBLEM (*)

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SOMMARIO. - *Si dà una semplice dimostrazione di esistenza ed unicità di soluzioni per una equazione differenziale*

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}(z_1, z_2) D_1^{\mu} D_2^{\nu} w + b(z_1, z_2) = 0$$

in due variabili complesse con opportuni coefficienti analitici.

SUMMARY. - *We give a simple proof for existence and uniqueness of solutions for a differential equation*

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}(z_1, z_2) D_1^{\mu} D_2^{\nu} w + b(z_1, z_2) = 0$$

in two complex variables with suitable analytic coefficients.

In the literature one may find many theorems about the Goursat problem for linear partial Differential equations with analytic coefficients (see e. g. Gilbert [1], Hörmander [2], Trèves [7], Vekua [8]). The most of them are in local form, but also for a wider class of equations and for special domains of regularity of the solutions, existence and uniqueness theorems for global solutions are proved.

In the present paper we give a simple proof for existence and uniquenes of solutions for a linear partial differential equation (1) in

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two complex variables with analytic coefficients specified below. We give a local proof by power series expansions due to Reich [5] and [6]. Then we extend the solutions by analytic continuation in a specified domain of $\mathbb{C} \times \mathbb{C}$, where the coefficients of the equation and the initial Goursat datas are holomorphic.

We start with some preliminary notations. Let G_1 and G_2 be simply connected open domains of $\mathbb{C}$ and let $G = G_1 \times G_2 \subset \mathbb{C}^2$.

Then $\mathbf{H}(G_1)$, $\mathbf{H}(G_2)$ and $\mathbf{H}(G)$ denote the sets of complex valued functions, holomorphic in G_1, G_2 respective G. Suppose that $(0, 0)$ lies in G.

Further we denote by

$$D_i := \frac{\partial}{\partial z_i}, \qquad i = 1, 2.$$

Now we may formulate our main theorem for the above specified domain G.

THEOREM: *Given an arbitrary function* $\varphi(z_1, z_2) \in \mathbf{H}(G)$ *and the differential equation*:

$$\text{(1)} \qquad \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}(z_1, z_2) D_1^{\mu} D_2^{\nu} w + b(z_1, z_2) = 0$$

$$a_{\mu\nu},\ b \in \mathbf{H}(G) \text{ and } a_{mn} \equiv 1.$$

Then there exists a unique solution $w \in \mathbf{H}(G)$, *satisfying the conditions*

$$\text{(2)} \qquad D_1^{\mu} D_2^{\nu} (w - \varphi) = 0 \text{ for } z_1 = 0 \text{ or } z_2 = 0;$$

$$\mu = 0, \dots, m-1; \qquad \nu = 0, \dots, n-1.$$

PROOF: We proceed in several steps:

1: First we give a proof of the local form of the Theorem. Let $P = (\xi, \eta)$ an arbitrary point of G and $\psi(z_1, z_2)$ any function holomorphic in P. Then we will show, that there is an unique solution w of (1), holomorphic in the point $P = (\xi, \eta)$ and satisfying the conditions:

$$\text{(2')} \qquad D_1^{\mu} D_2^{\nu} (w - \psi) = 0 \text{ for } z_1 = \xi \text{ or } z_2 = \eta;$$

$$\mu = 0, \dots, m-1; \qquad \nu = 0, \dots, n-1.$$

We introduce for w the power series expansion

$$w=\sum_{i,j\geq 0} w_{ij}(z_1-\xi)^i(z_2-\eta)^j. \tag{3}$$

By supposition, the coefficients of (1) have absolute convergent expansions in some neighbourhood of (ξ, η):

$$a_{\mu\nu}(z_1, z_2)=\sum_{i,j} a_{ij}^{\mu\nu}(z_1-\xi)^i(z_2-\eta)^j; \tag{4}$$

$$b(z_1, z_2)=\sum_{i,j} b_{ij}(z_1-\xi)^i(z_2-\eta)^j.$$

This together with (1) yields a recursiv formula for the coefficients w_{rs} with $r\geq m$ and $s\geq n$:

$$(m, r)(n, s)\, w_{rs}=-\sum_{\mu,\nu=0}^{m,n}\sum_{i,j\geq 0} a_{ij}^{\mu\nu}(\mu, r-m+\mu-i)(\nu, s-n+\nu-j)\cdot \tag{5}$$

$$\cdot w_{r-m+\mu-i,\, s-n+\nu-j}-b_{r-m,\, s-n}$$

where $(0, 1)=1$ and $(\mu, i)=i(i-1)\ldots(i-\mu+1)$ for $\mu>0$, and $a_{ij}^{mn}=0$ (by formal reason).

The coefficients w_{ij} of (3) with $0\leq i<m, j\geq 0$ or $i\geq 0, 0\leq j<n$ are exactly defined by (2), the others are determined recursively by (5). So the uniqueness of the local solution, depending on the local Goursat data ψ is guaranted.

We now prove the existence of the local solution by a majorization problem. Let

$$A_{\mu\nu}(z_1, z_2)=\sum_{i,j} A_{ij}^{\mu\nu}(z_1-\xi)^i(z_2-\eta)^j \tag{6}$$

and

$$B(z_1, z_2)=\sum_{i,j} B_{ij}(z_1-\xi)^i(z_2-\eta)^j$$

be a real majorant series of the series (4) and by formal reason $A_{mn}(z_1, z_2)\equiv 0$. Let also be

$$\Psi(z_1, z_2)=\sum_{i,j} \Psi_{ij}(z_1-\xi)^i(z_2-\eta)^j \tag{7}$$

a real majorant of the power series expansion of $\psi(z_1, z_2)$. Then we consider the solution of the equation

$$(8) \qquad W-\sum_{\mu,\nu=0}^{m,n}(z_1-\xi)^{m-\mu}\cdot(z_2-\eta)^{n-\nu}\cdot W\cdot A_{\mu\nu}-$$
$$-(z_1-\xi)^m\cdot(z_2-\eta)^n\cdot B-\Psi=0.$$

The holomorphic solution W is given by

$$(9) \qquad W=(\Psi+(z_1-\xi)^m\cdot(z_2-\eta)^n\cdot B)\cdot$$
$$\cdot\Big(1-\sum_{\mu,\nu=0}^{m,n}(z_1-\xi)^{m-\mu}\cdot(z_2-\eta)^{n-\nu}\cdot A_{\mu\nu}\Big)^{-1}$$

and, as one can easy see by computation of the power series expansion of W, this is a real majorant of (3). Thus w converges whenever W does converge.

So we may estimate the domain, where (3) is convergent.

Let $D_\rho(\xi)=\{z\in\mathbb{C}:|z-\xi|<\rho\}$ and $\rho\in\mathbb{R}$ be a positive number such, that the following conditions are satisfied for all $(z_1, z_2)\in D_\rho(\xi)\times D_\rho(\eta)$:

(i) the series (4) and (6) are absolutely convergent

(ii) the serie (7) is absolutely convergent

(iii) $1-\sum_{\mu,\nu=0}^{m,n}(z_1-\xi)^{m-\mu}(z_2-\eta)^{n-\nu}A_{\mu\nu}\neq 0.$

Then we know from (9) that the local solution $w(z_1, z_2)$ converges for all $(z_1, z_2)\in D_\rho(\xi)\times D_\rho(\eta)$. So the local part is proved.

Notice, that condition (2') is equivalent to the fact, that the uniqueness of the solution w is guaranted by prescribing the values:

$$D_1^\mu w(\xi, z_2)=f_\mu(z_2) \qquad \mu=0, 1, \dots, m-1$$
$$D_2^\nu w(z_1, \eta)=g_\nu(z_1) \qquad \nu=0, 1, \dots, n-1$$

for given $f_\mu(z_2)\in\mathbf{H}(D_\rho(\eta))$ and $g_\nu(z_1)\in\mathbf{H}(D_\rho(\xi))$, and that under these conditions $w\in\mathbf{H}(D_\rho(\xi)\times D_\rho(\eta))$.

2: We now come to the global problem. Let $K_1\subset G_1$, $K_2\subset G_2$ simply connected and compact, with $0\in K_i$ and set $K=K_1\times K_2$. Let $\rho\in\mathbb{R}$ now be a positive number, such that the conditions (i) and (iii) are satisfied for all $(\xi, \eta)\in K$ and also the power series expansions in each $(\xi, \eta)\in K$ of the initial data φ are convergent for $(z_1, z_2)\in D_\rho(\xi)\times D_\rho(\eta)$. We will show that, beginning with the local solution of (1) in $0=(0, 0)$, defined by (2), we can proceed by analytic continuation in a finite number of steps to the uniquely defined local solution in an arbitrary point $(\xi, \eta) \in K$ - denoted by $w(\xi, \eta; z_1, z_2)$ - whose domain of regularity contains $D_\rho(\xi) \times D_\rho(\eta)$. In fact we only have to show, that for each germ $w(\xi, \eta; z_1, z_2)$ the Goursat datas possess convergent power series expansions in $D_\rho(\xi) \times D_\rho(\eta)$.

2.1: Let $a, b \in \mathbb{C}$ such that:

(10) $$|a| < \rho, \ |b| < \rho \text{ and } |a+b| < \rho.$$

Let $(\xi_0, \eta_0) \in K$ and set $(\xi_1, \eta_1) = (\xi_0, \eta_0 + b)$, $(\xi_2, \eta_2) = (\xi_0 + a, \eta_0)$ and $(\xi_3, \eta_3) = (\xi_0 + a, \eta_0 + b)$. Suppose that all these points lie in K. We claim, that if for $\alpha = 0$, 1 and 2:

(11) $$w(\xi_\alpha, \eta_\alpha; z_1, z_2) = \sum_{i,j} w_{ij}^{(\alpha)} (z_1 - \xi_\alpha)^i (z_2 - \eta_\alpha)^j$$

converges for all $(z_1, z_2) \in D_\rho(\xi_\alpha) \times D_\rho(\eta_\alpha)$, then this is also true for $\alpha = 3$. Since $(\xi_3, \eta_3) \in D_\rho(\xi_\alpha) \times D_\rho(\eta_\alpha)$ for $\alpha = 0, 1, 2$, the serie (11) for $\alpha = 3$ is convergent in some neighbourhood of (ξ_3, η_3). We get

(12) $$w(\xi_3, \eta_3; z_1, z_2) = \sum_{i \geq 0} (z_1 - \xi_3)^i \frac{1}{i!} \frac{\partial^i}{\partial z_1^i} w(\xi_1, \eta_1; \eta_3, z_2)$$

and

(13) $$w(\xi_3, \eta_3; z_1, z_2) = \sum_{j \geq 0} (z_2 - \eta_3)^j \frac{1}{j!} \frac{\partial^j}{\partial z_2^j} w(\xi_2, \eta_2; z_1, \eta_3).$$

But the local Goursat datas ψ_3 of $w(\xi_3, \eta_3; z_1, z_2)$ are composed by

$$\psi_1 = \sum_{i=0}^{m-1} (z_1 - \xi_3)^i \frac{1}{i!} \frac{\partial^i}{\partial z_1^i} w(\xi_1, \eta_1; \xi_3, z_2)$$

and

$$\psi_2 = \sum_{j=0}^{n-1} (z_2 - \eta_3)^j \frac{1}{j!} \frac{\partial^j}{\partial z_2^j} w(\xi_2, \eta_2; z_1, \eta_3)$$

in that sense that $D_1^i(\psi_3 - \psi_1) = 0$ for $z_1 = \xi_3$, $i = 0, \dots, m-1$ and $D_2^j(\psi_3 - \psi_2) = 0$ for $z_2 = \eta_3$, $j = 0, \dots, n-1 \cdot \psi_1, \psi_2$ and therefore ψ_3 is convergent for $(z_1, z_2) \in D_\rho(\xi_3) \times D_\rho(\eta_3)$ and hence by the first part of the proof our claim is proved.

2.2. Suppose now that $(\xi, \eta) \in K$. Let γ be a curve in K, connecting $(0, 0)$ with (ξ, η). Let

$$(0, 0) = (\xi_0, \eta_0), (\xi_1, \eta_1), \dots, (\xi_r, \eta_r) = (\xi, \eta)$$

be a partition of this curve such, that all points of the closed curve segment between (ξ_i, η_i) and (ξ_{i+1}, η_{i+1}) lie in $D_\rho(\xi_{i+1}) \times D_\rho(\eta_{j+1})$. Then one sees easily by using the arguments of 2.1 and the fact, that $D_1^\mu w(0, y; 0, z_2) = D_1^\mu \varphi(0, z_2)$ and $D_2^\nu w(x, 0; z_1, 0) = D_2^\nu \varphi(z_1, 0)$ posses local power series expansions for each $x \in K_1$, $y \in K_2$ with radius of convergency ρ at least, that $w(x, y; z_1, z_2)$ converges in $D_\rho(x) \times D_\rho(y)$ for $x=0$ and $y=\eta_1, \dots, \eta_r$ and also for $y=0$ and $x=\xi_1, \dots, \xi_r$. So one can proceed step by step and show, that $w(x, y; z_1, z_2)$ converges in $D_\rho(x) \times D_\rho(y)$ also for $x=\xi_i$, $y=\eta_j$ with $i=1, \dots, r$ and $j=0, 1, \dots, r$.

One only has to notice, that the monodromy theorem holds (see e. g. Narasimhan [3], p. 18). Therefore we can state, that we have an analytic continuation of w along each curve γ to each $(\xi, \eta) \in K$ and the monodromy theorem again asserts, that we obtain a unique defined function w, holomorphic in each $(\xi, \eta) \in K$. Thus our theorem is proved.

As a conseguence one gets the following wellknown result, see Persson [4]:

COROLLARY: *Suppose that* $\varphi(z_1, z_2)$, $a_{\mu\nu}(z_1, z_2)$ *and* $b(z_1, z_2)$ *are entire functions, defined in* $\mathbb{C}^2$. *Then the unique solution* w *of* (1) *and* (2), *defined by the theorem is an entire function.*

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