EXISTENCE RELATIONS BETWEEN HARMONIC AND BIHARMONIC GREEN'S FUNCTIONS (*)

by MITSURU NAKAI and LEO SARIO (in Los Angeles) (**)

SOMMARIO. - *Su una varietà di Riemann siano β e γ due funzioni biarmoniche di Green caratterizzate dalle condizioni iniziali di Dirichlet $\beta = \partial\beta / \partial n = 0$ e $\gamma = \Delta\gamma = 0$ sul contorno ideale di M, e sia g la funzione armonica di Green su M. In questo lavoro ci occupiamo delle relazioni tra le classi O_β^N, O_γ^N, O_g^N di N-varietà di Riemann che non portano β, γ, g, rispettivamente.*

SUMMARY. - *On a Riemannian manifold M let β and γ be the biharmonic Green's functions characterized by Dirichlet data $\beta = \partial\beta / \partial n = 0$ and $\gamma = \Delta\gamma = 0$ at the ideal boundary of M, and let g the harmonic Green's function on M. In this paper we are interested in relations between the classes O_β^N, O_γ^N, O_g^N of Riemannian N-manifolds which do not carry β, γ, g, respectively.*

On a Riemannian manifold M, let β and γ be the biharmonic Green's functions characterized by « Dirichlet data $\beta=\partial\beta/\partial n=0$ and $\gamma=\Delta\gamma=0$ at the ideal boundary of M''. Denote by g the harmonic Green's function on M. We are interested in relations between the classes $O_\beta^N, O_\gamma^N, O_g^N$ of Riemannian N-manifolds which do not carry β, γ, or g, respectively.

It was shown in Ralston-Sario [7] that $O_\beta^N \subset O_\gamma^N$, and in Nakai-Sario [3] that, in fact, $O_\beta^N < O_\gamma^N$ for every $N \geq 2$. On the other hand,

(*) Pervenuto in Redazione il 28 febbraio 1976.
The work was sponsored by the U. S. Army Research Office, Grant DA-ARO-31-124-73-G 39, University of California, Los Angeles.
MOS Classification 31 B 30.

(**) Indirizzo degli Autori: Department of Mathematics - University of California - Los Angeles - California 90024 (U. S. A.).

$O_g^N < O_\gamma^N$, so that $O_\beta^N \cup O_g^N \subset O_\gamma^N$. Whether or not this inclusion is strict has been an open question. The main result of the present paper is that the inclusion *is* strict,

$$O_\beta^N \cup O_g^N < O_\gamma^N \qquad (N \geq 2),$$

i. e., there exist Riemannian manifolds of any dimension which carry both β and g but nevertheless fail to carry γ.

As to relations between O_β^N and O_g^N, it is known that $O_g^N - O_\beta^N \neq \emptyset$ (Nakai-Sario [2], [3]). We will show that

$$O_\beta^N \cap O_g^N \neq \emptyset, \qquad O_\beta^N - O_g^N \neq \emptyset \qquad (N \geq 2)$$

as well. That there also exist Riemannian N-manifolds ($N \geq 2$) carrying both β and g is trivial in view of the Euclidean N-ball.

1. Parabolic manifolds without β. We claim:

THEOREM 1. *There exist parabolic Riemannian manifolds of any dimension which carry no biharmonic Green's function* β,

$$O_\beta^N \cap O_g^N \neq \emptyset \qquad (N \geq 2).$$

PROOF. Let M be the N-space $\{0 \leq |x| = r < \infty\}$ with the metric

$$ds^2 = \varphi^2(r)\, dr^2 + \psi^{2/(N-1)}(r) \sum_{i=1}^{N-1} \lambda_i(\theta)\, (d\theta^i)^2,$$

where φ, ψ are strictly positive functions in $C^\infty[0, \infty)$ with $\varphi^2 = 1$, $\psi^{2/(N-1)} = r^2$ on $\{r < 1/2\}$ and the λ_i are the trigonometric functions of $\theta = (\theta^1, \dots, \theta^{N-1})$ which make the metric Euclidean on $\{r < 1/2\}$. Set

$$\sigma = \varphi\psi, \quad \tau = \varphi\psi^{-1}, \quad \lambda(\theta) = \left(\prod_{i=1}^{N-1} \lambda_i(\theta)^{1/2}\right).$$

In terms of the metric tensor, we have $g^{1/2} = \sigma\lambda$ and $g^{1/2} g^{rr} = \tau^{-1}\lambda$. The Laplace-Beltrami operator is

$$\Delta = d\delta + \delta d = -\, g^{-1/2} \left[\frac{\partial}{\partial r}\left(g^{1/2} g^{rr} \frac{\partial}{\partial r}\right) + \sum_{i=1}^{N-1} \frac{\partial}{\partial \theta^i}\left(g^{1/2} g^{ii} \frac{\partial}{\partial \theta^i}\right)\right],$$

where we have used the Einstein summation convention.

The function

$$h(r) = \int_1^r \tau(s)\, ds$$

satisfies the harmonic equation

$$\Delta h(r) = -\sigma^{-1}(\tau^{-1} h')' = 0.$$

For a fixed $\rho \in (0, \infty)$, the function

$$q_\rho(r) = \int_r^\rho \tau \int_1^t \sigma\, ds\, dt$$

satisfies the quasiharmonic equation

$$\Delta q(r) = -\sigma^{-1}(\tau^{-1} q')' = 1,$$

and the function

$$u_\rho(r) = \int_r^\rho \tau \int_1^v \sigma \int_1^t \tau\, ds\, dt\, dv$$

satisfies the biharmonic equation

$$\Delta u(r) = -\sigma^{-1}(\tau^{-1} u')' = \int_1^r \tau(s)\, ds.$$

The function

$$\beta_\rho(r) = u_\rho(r) + c_\rho q_\rho(r), \quad c_\rho = -\frac{u'_\rho(\rho)}{q'_\rho(\rho)},$$

is biharmonic and meets the boundary conditions

$$\beta_\rho(\rho) = \beta'_\rho(\rho) = 0.$$

We write in extenso

$$\beta_\rho(r) = \int_r^\rho \tau \int_1^v \sigma \int_1^t \tau\, ds\, dt\, dv - \frac{\int_1^\rho \sigma \int_1^t \tau\, ds\, dt}{\int_1^\rho \sigma\, ds} \int_r^\rho \tau \int_1^t \sigma\, ds\, dt.$$

On $\{r>1\}$, choose $\sigma=\tau=1$, i. e., take the metric

$$ds^2=dr^2+\sum_{i=1}^{N-1}\lambda_i(\theta)(d\theta^i)^2.$$

Then

$$h(r)=\int_1^r ds$$

is unbounded, hence the harmonic measure of the ideal boundary $r=\infty$ of M on $\{r\geq 1\}$ is $\omega=ah+b=b=$ const, and we have $M\in O_g^N$ (cf., e. g., Sario-Nakai [10]). On the other hand,

$$\beta_\varrho(r)=\int_r^\varrho\int_1^v(t-1)\,dt\,dv-\frac{\int_1^\varrho(t-1)\,dt}{\rho-1}\int_r^\varrho(t-1)\,dt=$$

$$=\frac{1}{6}[(\rho-1)^3-(r-1)^3]-\frac{1}{4}(\rho-1)\,[(\rho-1)^2-(r-1)^2]$$

is unbounded in ρ. Since the existence of β on M is independent of the pole and the exhaustion (Nakai-Sario [4]), we conclude that $M\in O_\beta^N$.

2. Hyperbolic manifolds without β. It is known that there exist parabolic manifolds which carry β (Nakai-Sario [2], [3]). We proceed to show:

THEOREM 2. *There exist hyperbolic Riemannian manifolds of any dimension which carry no biharmonic Green's functions* β,

$$O_\beta^N-O_g^N\neq\varnothing\qquad(N\geq 2).$$

PROOF. This is a corollary of $O_\beta^N-O_{HD}^N\neq\varnothing$ established in Nakai-Sario [6]. Here we give an independent proof. Consider again the N-space with the metric

$$ds^2=\varphi(r)^2\,dr^2+\psi(r)^{2/(N-1)}\sum_{i=1}^{N\ 1}\lambda_i(\theta)(d\theta^i)^2,$$

but now take

$$h(r)=\int_r^\infty \tau(s)\,ds$$

and

$$\beta_\varrho(r)=\int_r^\varrho \tau \int_1^v \sigma \int_t^\infty \tau\,ds\,dt\,dv-\frac{\int_1^\varrho \sigma \int_t^\infty \tau\,ds\,dt}{\int_1^\varrho \sigma\,ds}\int_r^\varrho \tau \int_1^t \sigma\,ds\,dt.$$

On $\{r>1\}$, choose $\sigma=r$, $\tau=r^{-2}$, i. e., consider the metric

$$ds^2=r^{-1}\,dr^2+r^{3/(N-1)}\sum_{i=1}^{N-1}\lambda_i(\theta)\,(d\theta^i)^2.$$

Then

$$h(r)=\int_r^\infty s^{-2}\,ds$$

is bounded, hence $M\notin O_g^N$, whereas

$$\beta_\varrho(r)=\int_r^\varrho v^{-2}\int_1^v t\cdot t^{-1}\,dt\,dv-\frac{\rho-1}{(1/2)(\rho^2-1)}\int_r^\varrho t^{-2}\frac{1}{2}(t^2-1)\,dt=$$

$$=\log\frac{\varrho}{r}+(\rho^{-1}-r^{-1})-\frac{1}{\rho+1}(\rho-r+\rho^{-1}-r^{-1})$$

is unbounded in ρ, and we have $M\in O_\beta^N$.

3. Hyperbolic manifolds with β but without γ. The goal of the remainder of the present work is to show that, for $N\geq 2$,

(1) $$O_\gamma^N-O_\beta^N\cup O_g^N\neq\varnothing.$$

For $N=2$, the Euclidean half-plane belongs to this class, as was first shown by J. Ralston. The authors are pleased to acknowledge the gratitude to Professor Ralston for communicating this unpublished result to them. Ralston's elegant proof is based on an explicit formula for β in the case $N=1$ and a further development of the technique in Ralston-Sario [7]. Here we give an alternate proof which utilizes results in Nakai-Sario [4]. Let Π^2 be the half-plane $x^1>0$ in E^2. The harmonic Green's function

$$g(z,\zeta)=\frac{1}{2\pi}\log\left|\frac{z+\bar{\zeta}}{z-\zeta}\right|$$

on Π^2 gives trivially $\Pi^2\notin O_g^2$. Let Ω_ζ be a neighborhood of ζ with $\bar{\Omega}_\zeta\subset\Pi^2$, and set $G=\Pi^2-\bar{\Omega}_\zeta$, $G_\rho=\{r>\rho\mid |\arg z|<\pi/4\}\cap G$. As $\rho\to\infty$,

$$\|g\|_G^2=c\int_G\left|\log\left|\frac{z+\bar{\zeta}}{z-\zeta}\right|\right|^2 r\,dr\,d\theta\geq c\int_{G_\rho}\left|\log\left|\frac{z-\bar{\zeta}}{z-\zeta}\right|^2\right|r\,dr\,d\theta\sim$$

$$\sim c\int_\rho^\infty\frac{1}{r^2}\cdot r\,dr=\infty,$$

hence $\Pi^2\in O_\gamma^N$. On the other hand, we showed in [4] that the subregion $\Sigma^2=\{|z|>1\}$ of E^2 carries β. Since the existence of β on a Riemannian manifold entails that on a subregion (loc. cit.), the relation Π^2+ $+1\subset\Sigma^2\notin O_\beta^2$ gives $\Pi^2+1\notin O_\beta^2$ and, therefore, $\Pi^2\notin O_\beta^2$.

Actually, for $\Sigma^N=\{r>1\}$ in E^N, we proved in [4] that

$$\Sigma^N\in O_\gamma^N-O_\beta^N\cup O_g^N\qquad(N=2,3,4).\tag{2}$$

This example has the virtue of being simple and natural. However, since $E^N\notin O_\gamma^N\cup O_\beta^N\cup O_g^N$ for $N\geq5$, every subregion of E^N has the same property. As a consequence, there do not exist « simple and natural » Riemannian manifolds in $O_\gamma^N-O_\beta^N\cup O_g^N$ for $N\geq5$. That this class is, nevertheless, nonvoid for $N\geq5$ as well is the main result of the present paper.

4. A test for $O_\gamma^N-O_\beta^N\cup O_g^N$. Our construction will be guided by the following test, a direct consequence of our results in [4]. On a Riemannian N-manifold $M\notin O_\beta^N$ with exhausting regular subregions Ω, we continue referring to the uniform convergence of β_Ω to β_M on compact subsets of $M\times M$ as the *consistency* of β_M on $M\times M$. The Riemannian volume element at $x\in M$ is denoted by dV_x.

THEOREM 3. *Let M be a hyperbolic Riemannian N-manifold* $(N\geq 2)$ *with the harmonic Green's function* $G_M(x, y)$. *For a parametric ball B with center* η, *suppose*

$$\int_{M-\bar{B}} G_M(x, \eta)^2 \, dV_x = \infty \tag{3}$$

but

$$\int_{M-\bar{\Omega}} (G_M(x, y) - G_M(x, \eta))^2 \, dV_x < \infty \tag{4}$$

for any $y\in M-\bar{B}$ *and any regular subregion* Ω *of* M *with* $\Omega\supset\bar{B}\cup y$. *Then*

$$M-\bar{B}\in O_\gamma^N - O_\beta^N \cup O_g^N, \tag{5}$$

and $\beta_{M-\bar{B}}$ *is continuous and consistent on* $(M-\bar{B})\times(M-\bar{B})$.

The relations $M-\bar{B}\notin O_g^N$ and $M-\bar{B}\in O_\gamma^N$ are again immediate. The function $G_M(x, y) - G_M(x, \eta)$ is a fundamental kernel on $M-\bar{B}$ in the sense of [4], and a fortiori $M-\bar{B}\notin O_\beta^N$. If, in the definition of O_β^N, we disregard the continuity and consistency of β on the product space, then, $G_M(x, y) - G_M(x, \eta)$ being square integrable off its pole y, the characterization of O_β^N given in [5] makes the relation $M-\bar{B}\notin O_\beta^N$ as trivial as $M-\bar{B}\notin O_\gamma^N$.

5. Comparison principle. We insert here a general statement which will be used later. Let $0\leq\alpha<\beta<\infty$ and $a\in C^1(\alpha, \beta)$. Consider the ordinary differential operator

$$Lu = (au')' - pu \tag{6}$$

with $p\in C(\alpha, \beta)$. If a function u satisfies

$$Lu \leq 0 \tag{7}$$

on (α, β), then it is called a *supersolution* on (α, β). If a supersolution u satisfies

$$\liminf_{r\to\alpha} u(r) \geq 0, \qquad \liminf_{r\to\beta} u(r) \geq 0, \tag{8}$$

then $u \geq 0$ on (α, β). This result was obtained, and called the *comparison principle,* by Nakai [1] (the proof for the above operator is the same as for the elliptic operator). We will use this principle in the following form:

LEMMA. *Let u be a solution of $Lu=0$ in (α, β) with boundary values $u(\alpha)$ and $u(\beta)$, and let v be a supersolution, i. e., $Lv \leq 0$, on (α, β) with boundary values $v(\alpha)=u(\alpha)$ and $v(\beta)=u(\beta)$. Then $u \leq v$ on (α, β).*

6. Expansions in spherical harmonics. For convenience, we compile here some fundamentals on spherical harmonics. At a point $x=(x^1, \dots, x^N)$ of E^N, $N \geq 2$, the line element $|dx|^2 = \Sigma_{i=1}^N (dx^i)^2$ reads in polar coordinates $(r, \theta)=(r, \theta^1, \dots, \theta^{N-1})$

(9) $$|dx|^2 = dr^2 + r^2 \sum_{i=1}^{N-1} \lambda_i(\theta)(d\theta^i)^2$$

with $|x|=r$ and the λ_i certain trigonometric functions of $\theta=(\theta^1, \dots, \theta^{N-1})$. The surface element $d\theta$ of the unit sphere $\Theta_N = \{ |x| = r = 1 \}$ is

(10) $$d\theta = \lambda(\theta)\, d\theta^1 \dots d\theta^{N-1}, \quad \lambda(\theta) = \left(\prod_1^{N-1} \lambda_i(\theta) \right)^{1/2},$$

the area ω_N of Θ_N is

(11) $$\int_\Theta d\theta = \omega_N = 2\pi^{N/2} (\Gamma(N/2)^{-1},$$

and the volume element is $dx = dx^1 \dots dx^N = r^{N-1}\, dr\, d\theta$. For the Euclidean Laplace-Beltrami operator Δ_N we have

(12) $$\Delta_N \varphi = -r^{-(N-1)} \frac{\partial}{\partial r}\left(r^{N-1} \frac{\partial}{\partial r} \varphi \right) - \\ - (\lambda(\theta) r^2)^{-1} \sum_{i=1}^{N-1} \frac{\partial}{\partial \theta^i} \left(\lambda(\theta) \lambda_i(\theta)^{-1} \frac{\partial}{\partial \theta^i} \varphi \right).$$

A spherical harmonic $S_n(\theta)$ of degree $n \geq 1$ is, by definition, characterized by $\Delta_N(r^n S_n) = 0$, and therefore,

(13) $$\Delta_N S_n = n(n+N-2) r^{-2} S_n.$$

Let $\{S_{nm}\}$ $(m=1, \dots, m_n)$ be the complete orthonormal system of spherical harmonics of degree $n \geq 1$. Then

$$\left\{ \frac{1}{\sqrt{\omega_N}},\ S_{nm};\ m=1, \dots, m_n,\ n=1, 2, \dots \right\}$$

is a complete orthonormal system in $L_2(\Theta, d\theta)$. Moreover, if $\varphi \in C^1(\Theta)$, then

$$\varphi = c_0 + \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} c_{nm} S_{nm}$$

with $c_0 = (\varphi, 1)/\omega_N$, $c_{nm} = (\varphi, S_{nm})$, the inner product being in $L_2(\Theta, d\theta)$, and the series is absolutely and uniformly convergent on Θ; if φ depends on a parameter $r \in [r_1, r_2]$ and $\varphi \in C^1([r_1, r_2] \times \Theta)$, then the convergence is uniform on $[r_1, r_2] \times \Theta$.

7. Main result. We endow $\Sigma^N = \{ |x| = r > 1 \}$ with the metric

$$ds^2 = r^4\, dr^2 + r^{8/(N-1)} \sum_{1}^{N-1} \lambda_i(\theta)\, (d\theta^i)^2, \tag{14}$$

where the $\lambda_i(\theta)$ $(i=1, \dots, N-1)$ are as in (9), and we denote by Σ_{ds}^N the resulting Riemannian manifold. We maintain:

MAIN THEOREM. *The manifold Σ_{ds}^N $(N \geq 5)$ is hyperbolic, carries no γ, but carries a β which is continuous and consistent on $\Sigma_{ds}^N \times \Sigma_{ds}^N$:*

$$\Sigma_{ds}^N \in O_\gamma^N - O_\beta^N \cup O_g^N \qquad (N \geq 5). \tag{15}$$

The proof will be given in Nos. 7-11.

Choose strictly positive C^∞ functions $R_1(r)$, $R_2(r)$ on $[0, \infty)$ such that $R_1(r) = r^4$, $R_2(r) = r^{8/(N-1)}$ on $[1, \infty)$ and $R_1(r) = 1$, $R_2(r) = r^2$ on $[0, 1/2]$. The metric

$$d\tilde{s}^2 = R_1(r)\, dr^2 + R_2(r) \sum_{1}^{N-1} \lambda_i(\theta)\, (d\theta^i)^2,$$

is C^∞ on E^N, $d\tilde{s} = ds$ on Σ^N, and $d\tilde{s} = |dx|$ on $|x| < 1/2$. Accordingly, we may and will henceforth view ds as *a metric on* E^N, with $ds = |dx|$ on $|x| < 1/2$.

8. Hyperbolicity. In the metric ds on E^N, the volume element dV on Σ^N is

(16) $$dV=r^6\,dr\,d\,\theta,$$

the surface element dS on $|x|=r\geq 1$ is

(17) $$dS=r^4\,d\,\theta,$$

and the interior normal derivative $\partial\varphi/\partial n$ on $|x|=r>1$ is

(18) $$\frac{\partial\varphi}{\partial n}=-r^{-2}\frac{\partial\varphi}{\partial r}.$$

The Laplace-Beltrami operator Δ with respect to ds takes the form

(19) $$\Delta\varphi=-r^{-6}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\varphi\right)-$$
$$-(\lambda(\theta)\,r^{8/(N-1)})^{-1}\sum_1^{N-1}\frac{\partial}{\partial\theta^i}\left(\lambda(\theta)\,\lambda_i(\theta)^{-1}\frac{\partial}{\partial\theta^i}\varphi\right).$$

For a function $\psi(\theta)$, (12) and (19) give

(20) $$\Delta\psi=r^{2-8/(N-1)}\Delta_N\psi\quad(\psi=\psi(\theta)).$$

As a counterpart of (13), we thus have

(21) $$\Delta S_n=n(n+N-2)\,r^{-8/(N-1)}\,S_n$$

for every spherical harmonic S_n of degree $n\geq 1$.

For a function $\psi(r)$,

(22) $$\Delta\psi=-r^{-6}\frac{d}{dr}\left(r^2\frac{d}{dr}\psi\right)\quad(\psi=\psi(r)).$$

In terms of the ordinary differential operator

$$L\psi=\frac{d}{dr}\left(r^2\frac{d}{dr}\psi\right),$$

$\psi(x)=\psi(|x|)$ belongs to the class $H(|x|>1)$ of harmonic functions relative to ds if and only if $L\psi=0$, i. e.,

$$\psi = c_0 + c_1 r^{-1},$$

with c_0, c_1 constants. Thus

$$1 - \rho/|x| = 1 - \rho/r$$

is the *harmonic measure* of the ideal boundary ∞ of Σ_{ds}^N (and of E_{ds}^N) on $|x| > \rho \geq 1$. Therefore, $E^N \notin O_g^N$ and

(23) $$\Sigma^N \notin O_g^N.$$

9. An inequality. The constant $\rho_N = N^{(N-1)/(6N-14)}$ dominates 1 in our case $N \geq 5$. This ρ_N is so chosen that

(24) $$n + N - 3 \leq n r^{(6N-14)/(N-1)}$$

for every $n = 1, 2, \ldots$ and every $r \in [\rho_N, \infty)$. Consider the ordinary differential operator

(25) $$L_n \psi = \frac{d}{dr}\left(r^2 \frac{d}{dr}\psi\right) - n(n+N-2)\, r^{(6N-14)/(N-1)}\, \psi$$

for each $n = 0, 1, 2 \ldots$ on $[1, \infty)$. Observe that

$$L_n r^{-1} = -n(n+N-2)\, r^{(6N-14)/(N-1)}\, r^{-1} \leq 0,$$

i. e., r^{-1} is a supersolution of $L_n \psi = 0$ on $[1, \infty)$. Since 0 is a solution of $L_n \psi = 0$, the Perron method assures the *unique existence* of a solution u of $L_n \psi = 0$ on $[1, \infty)$ with boundary values $u(1) = 1$ and $u(\infty) = 0$. Hence there exists a *unique solution*

$$e_n(r; \rho)$$

of $L_n \psi = 0$ on $[\rho, \infty)$ $(\rho \geq 1)$ with boundary values $e_n(\rho; \rho) = 1$ and $e_n(\infty; \rho) = 0$. The key relation in our reasoning will be

(26) $$o < e_n(r; \rho) \leq \rho^{n+N-2}/r^{n+N-2}$$

for every $n = 1, 2, \ldots$ and $r \in [\rho, \infty)$, with $\rho \geq \rho_N$. For the proof, (24) gives

$$L_n r^{-n-N+2} = (n+N-2)\left(n+N-3-n r^{(6N-14)/(N-1)}\right) r^{-n-N+2} \leq 0.$$

By the comparison principle, we obtain (26).

10. Fourier expansion. Let $h \in H(|x| > \rho)$ $(\rho \geq \rho_N)$. We consider its Fourier expansion

(27) $$h(r, \theta) = h_0(r) + \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} h_{nm}(r) S_{nm}(\theta)$$

for $r \in (\rho, \infty)$, with $\Delta h_0 = \Delta (h_{nm} S_{nm}) = 0$ for every n and m. The expansion converges absolutely and uniformly on compact sets of $\{|x| > \rho\}$. By (19) and (21),

(28) $$L_0 h_0 = L_n h_{nm} = 0$$

for every n and m on (ρ, ∞). We assume that $h(r, \theta)$ is continuous on $\rho \leq |x| \leq \infty$ and of class C^1 on $\rho \leq |x| < \infty$, and

(29) $$\lim_{r \to \infty} h(r, \theta) = 0.$$

Then h_0 and h_{nm} also are continuous on $[\rho, \infty]$ and

(30) $$h_0 = c_0 e_0(\cdot; \rho), \quad h_{nm} = c_{nm} e_n(\cdot; \rho),$$

with $c_0 = h_0(\rho)$, $c_{nm} = h_{nm}(\rho)$, and

(31) $$|c_0| + \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} |c_{nm}| < \infty, \qquad |c_0|^2 + \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} |c_{nm}|^2 < \infty.$$

Expression (27) thus takes the form

(32) $$h(r, \theta) = c_0 e_0(r; \rho) + \sum_{n=1}^{\infty} \left(\sum_{m=1}^{m_n} c_{nm} S_{nm}(\theta) \right) e_n(r; \rho)$$

on $[\rho, \infty]$.

11. Conclusion. Let $G(x, y) = G(r, \theta; y)$ be the harmonic Green's function on E_{ds}^N, normalized by the flux $\int_\alpha * dG = -1$ across a hypersphere α about y. In particular,

(33) $$G(r, \theta; 0) = c e_0(r; 1) = c r^{-1}$$

with $c=G(1,\theta;0)$, on Σ_{ds}^N. By (16),

$$\int_{\Sigma_{ds}^N} G(x,0)^2\,dV_x=\omega_N c\int_1^\infty r^{-2}\cdot r^6\,dr=\infty. \tag{34}$$

We claim that

$$\int_{|x|>\rho} (G(x,y)-G(x,0))^2\,dV_x<\infty \quad (\rho=\rho_N+|y|) \tag{35}$$

for $y\in\Sigma_{ds}^N$. For the proof, we apply (32) to $G(\cdot,z)$ with $|z|\leq|y|$:

$$G(r,\theta;z)=c_0(z)\,e_0(r;\rho)+\sum_{n=1}^{\infty}\left(\sum_{m=1}^{m_n} c_{nm}(z)\,S_{nm}(\theta)\right)e_n(r;\rho) \tag{36}$$

for $r\geq\rho$. Since

$$c_0(z)\,e_0(r;\rho)=\frac{1}{\omega_N}\int_\Theta G(r,\theta;z)\,d\theta,$$

we have

$$c_0(z)\frac{d}{dr}e_0(r;\rho)=\frac{1}{\omega_N}\int_\Theta \frac{d}{dr}G(r,\theta;z)\,d\theta,$$

and, by (17) and (18), obtain

$$-c_0(z)\,r^2\frac{d}{dr}e_0(r;\rho)=\frac{1}{\omega_N}\int_{|x|=r}\frac{\partial}{\partial n_x}G(x;z)\,dS.$$

Here the integral is the value at z of the harmonic function on $|x|<r$ with boundary values 1 on $|x|=r$, i. e., of the constant function 1. Since $e_0(r;\rho)=\rho/r$,

$$c_0(z)=(\omega_N\,\rho)^{-1}, \tag{37}$$

i. e., $c_0(z)$ is constant for all $|z|\leq|y|$. Therefore,

$$G(r,\theta;y)-G(r,\theta;0)=\sum_{n=1}^{\infty}\left(\sum_{m=1}^{m_n} d_{nm}\,S_{nm}(\theta)\right)e_n(r;\rho) \tag{38}$$

where $d_{nm}=c_{nm}(y)-c_{nm}(0)=c_{nm}(y)$. Thus,

$$\int_{\Theta}(G(r,\theta;y)-G(r,\theta;0))^2\,d\theta=\sum_{n=1}^{\infty}\left(\sum_{m=1}^{m_n}d^2_{nm}\right)e_n(r;\rho)^2$$

and consequently,

$$\int_{|x|>\varrho}(G(x,y)-G(x,0))^2\,dV_x=\sum_{n=1}^{\infty}k_n\int_{\varrho}^{\infty}e_n(r;\rho)^2\,r^6\,dr \tag{39}$$

with $k_n=\sum_{m=1}^{m_n}d^2_{nm}$. Here, by (31),

$$\sum_{n=1}^{\infty}k_n<\infty. \tag{40}$$

Now we make use of (26):

$$\int_{\varrho}^{\infty}e_n(r;\rho)^2\,r^6\,dr\le\rho^{2(n+N-2)}\int_{\varrho}^{\infty}r^{-2n-2N+4}\cdot r^6\,dr\le \tag{41}$$

$$\le\rho^{2(n+N-2)}\int_{\varrho}^{\infty}r^{-2n}\,dr\le$$

$$\le\frac{1}{2n-1}\,\rho^{2N-3}$$

since $N\ge5$. By (39),

$$\int_{|x|>\varrho}(G(x,y)-G(x,0))^2\,dV_x\le\sum_{n=1}^{\infty}\rho^{2N-3}\cdot\frac{k_n}{2n-1}\,. \tag{42}$$

This with (40) implies (35).

With (34), (35), and Theorem 3, the proof of the Main Theorem is complete.

BIBLIOGRAPHY

[1] M. NAKAI, *A test of Picard's principle for rotation free density,* J. Math. Soc. Japan 27 (1975), 412-431.

[2] M. NAKAI - L. SARIO, *Parabolic Riemannian planes carrying biharmonic Green's functions of the clamped plate,* J. Analyse Math. 30 (1976), 372-389.

[3] M. NAKAI - L. SARIO, *A strict inclusion related to biharmonic Green's functions of clamped and simply supported bodies,* Ann. Acad. Sci. Fenn. A. I. 3 (1977), 53-58.

[4] M. NAKAI - L. SARIO, *Existence of biharmonic Green's functions,* Proc. London Math. Soc. (3) 36 (1978).

[5] M. NAKAI - L. SARIO, *A nonexistence test for biharmonic functions of clamped bodies,* Math. Scand. 40 (1977) 151-160.

[6] M. NAKAI - L. SARIO, *Manifolds with strong harmonic boundaries but without Green's functions of clamped bodies,* Ann. Scuola Norm. Sup. Pisa 3 (1976), 665-670.

[7] J. RALSTON - L. SARIO, *A relation between biharmonic Green's functions of simply supported and clamped bodies,* Nagoya Math. J. 61 (1976), 59-71.

[8] L. SARIO, *A criterion for the existence of biharmonic functions,* J. Austral. Math. Soc. 21 (1976), 155-165.

[9] L. SARIO, *Biharmonic Green's functions and harmonic degeneracy,* J. Math. Kyoto Univ. 15 (1975), 351-362.

[10] L. SARIO - M. NAKAI, *Classification Theory of Riemann Surfaces,* Grundlagen der Mathematischen Wissenschaften 164, 1970, 446 pp.

[11] C. WANG, *Biharmonic Green's functions and quasiharmonic degeneracy,* Math. Scand. 35 (1974), 38-42.

[12] C. WANG, *Biharmonic Green's functions and biharmonic degeneracy,* Math. Scand. 37 (1975), 122-128.