ON THE METRIC ENTROPY OF SOME CLASSES OF HOLOMORPHIC FUNCTIONS (*)

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SOMMARIO. - *Si determina la ε-entropia e la ε-capacità (nel senso di Kolmogorov) per funzioni con integrale limitato di Dirichlet e per funzioni « mean p-valent ».*

SUMMARY. - *The ε-entropy and ε-capacity (in the sense of Kolmogorov) of functions with bounded Dirichlet integral and « mean p-valent » functions are determined.*

To characterize the « massiveness » of a totally bounded set A in the metric space R A. N. Kolmogorov introduced the functions $H_\varepsilon(A)$ (metric or ε-entropy) and $C_\varepsilon(A)$ (ε-capacity) which are defined to be the dual logarithm ($ld\,x$) of the minimal number of sets in an ε-covering of A and of the minimal number of points in an ε-net for the set A respectively. In this note we want to determine the asymptotic behaviour of the functions H_ε and C_ε for two classes of holomorphic functions endowed with various norms, with methods established by Vituskin and Erohin [3].

First we investigate the set $\mathcal{D}$ of functions $f(z)=\sum_{n=0}^{\infty} a_n z^n$ with bounded Dirichlet integral (1):

$$(1) \qquad M(\mathcal{D}, f)=\int_{-\pi}^{\pi}\int_{0}^{1} |f'(re^{i\theta})|^2\, r\, dr\, d\theta=\sum_{n=1}^{\infty} n\,|a_n|^2<\infty .$$

(*) Pervenuto in Redazione il 28 febbraio 1976.
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Since $\mathcal{D}$ is contained in the Hardy class H^2, we use for a fixed r $(0<r<1)$ the norm

$$||f_r||_2=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^2\,d\theta\right)^{1/2}.$$

THEOREM 1: *For a fixed* r $(0<r<1)$ *we have for the set* $\mathcal{D}$ *of functions with bounded Dirichlet integral in the normed space* $(H^2, ||\cdot_r||_2)$ *the asymptotic formula*

$$H_\varepsilon(\mathcal{D})|=C_\varepsilon(\mathcal{D})=\frac{\left(ld\frac{1}{\varepsilon}\right)^2}{ld\frac{1}{r}}+0\left(ld\frac{1}{\varepsilon}\,ld\,ld\frac{1}{\varepsilon}\right).$$

PROOF. Let $f(z)=\sum_{n=0}^{\infty}a_n z^n=\sum_{n=0}^{\infty}a_n r^n e^{in\theta}=\sum_{n=0}^{\infty}c_n e^{in\theta}$ be a function of $\mathcal{D}$. With Hölder's inequality we obtain

$$|c_n|=\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}f(re^{i\theta})\,e^{-in\theta}\,d\theta\right|\leqq||f_r||_2=\left(\sum_{n=0}^{\infty}|c_n|^2\right)^{1/2}\leqq C_0\sum_{n=0}^{\infty}|c_n|$$

and thus

(2) $$\sup_{n\geqq 0}|c_n|\leqq||f_r||_2\leqq C_0\sum_{n=0}^{\infty}|c_n|.$$

Because of (1) we have $|a_n|\leqq C'$ and hence

(3) $$|c_n|=r^n|a_n|\leqq C'e^{-n\log(1/r)}.$$

Given $\Delta>0$ we choose C_1 so that $\frac{C_1^2\cdot e^{-2\Delta}\cdot\Delta^2}{(1-e^{-2\Delta})^2}\leqq C$ for some given positive costant C. If

(4) $$|c_n|\leqq C_1 e^{-n(\log(1/r)+\Delta)},$$

we obtain

$$\sum_{n=1}^{\infty}n|a_n|^2=\sum_{n=1}^{\infty}nr^{-2n}|c_n|^2\leqq C_1^2\cdot\Delta^2\sum_{n=1}^{\infty}ne^{-2n\Delta}\leqq C.$$

Hence $f(z)\in\mathcal{D}(C)=\{f\in\mathcal{D}: M(\mathcal{D},f)\leqq C\}\subset\mathcal{D}$.

Because of (2), (3) and (4) theorem XVII of [3] can be applied. This yields, together with the inequalities $C_{2\varepsilon} \leqq H_{\varepsilon} \leqq C_{\varepsilon}$ and the semiadditivity of C_ε and H_ε, our theorem.

In the second part of this note we investigate the set of functions mean p-valent ($p>0$) in the unit disc D. Let $n(r, w)$ be the number of roots in $D_r = \{z: |z| < r\}$ of the equation $f(z) = w$. We write

$$P(r, \rho) = \frac{1}{2\pi} \int_{-\pi}^{\pi} n(r, \rho e^{i\theta})\, d\theta$$

and

$$W(r, R) = \int_0^R p(r, \rho)\, d\rho^2 = \frac{1}{\pi} \int_0^R \int_{-\pi}^{\pi} n(r, \rho e^{i\theta})\, \rho\, d\rho\, d\theta.$$

Following Spencer we shall call a function $f(z)$ mean p-valent in the unit disc D if $f(z)$ is regular in$D$, and

$$W(1, R) = W(R) \leqq p R^2 \qquad (0 < R < \infty).$$

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we write

$$A_p = \max_{0 \leqq \nu \leqq [p]} |a_\nu|.$$

Let us consider the following classes of functions (see to this respect also [1]):

$$M_p(C) = \{f: f \text{ mean } p\text{-valent in } D \text{ and } A_p \leqq |C|\}$$

$$M_p^*(C) = \{f: f(z) = C + \sum_{n=1}^{\infty} a_n z^n, \quad \sum_{n=1}^{\infty} n |a_n|^2 \leqq p |C|^2/4$$

$$\text{and } \sum_{n=1}^{\infty} |a_n| \leqq |C|/2 \}$$

with some complex constant $C \neq 0$. Analogous to [3] we write $c_n = r^n a_n$ and obtain from Cauchy's inequality that

$$\sup_{n \geqq 0} |c_n| \leqq ||f||_r \leqq C' \sum_{n=0}^{\infty} |c_n| \tag{5}$$

with $||f||_r = \sup_{|z| \leqq r} |f(z)|$.

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a function of $M_p(C)$ and $\alpha > 0$, theorem 3.5 of [2] yields, after short computations,

$$|c_n| = |r^n a_n| \leqq C_2(p, \alpha) \cdot e^{-n(\log(1/r) - \alpha)} \quad n = 1, 2, \dots \tag{6}$$

with

$$C_2(p, \alpha) = \begin{cases} |C|\, C_1(p) \left(\dfrac{2p-1}{\alpha e}\right)^{2p-1} & (p > 1/2) \\ |C|\, C_1(p)\, e^{-\alpha} & (0 < p \leqq 1/2). \end{cases}$$

Because of (5) and (6) we obtain from [3], p. 319 the inequality

$$H_\varepsilon(M_p(C)) \leqq \frac{\left(ld \dfrac{1}{\varepsilon}\right)^2}{ld\, e \left(\log \dfrac{1}{r} - \alpha\right)} + 0\left(ld \frac{1}{\varepsilon}\, ld\, ld \frac{1}{\varepsilon}\right). \tag{7}$$

Now we choose a constant $C_3(p, \alpha, C)$ so that (i) and (ii) is valid:

$$C_3^2(p, \alpha, C) \cdot \sum_{n=1}^{\infty} n \cdot e^{-2\alpha n} \leqq p\, |C|^2/4 \tag{i}$$

$$C_3(p, \alpha, C) \cdot (e^\alpha - 1)^{-1} \leqq |C|/2. \tag{ii}$$

Assume that for $\alpha > 0$ and $p > 0$, (5) and

$$|c_n| \leqq C_3(p, \alpha, C) \cdot e^{-n(\log(1/r) + \alpha)} \quad n = 1, 2, \dots \tag{8}$$

is satisfied. Because of (i) and (ii) $f(z) = C + \sum_{n=1}^{\infty} c_n r^{-n} z^n$ is a function of class $M_p^*(C)$.

Now we prove that $M_p^*(C)$ is a subset of $M_p(C)$. For $f(z) \in M_p^*(C)$ we have $a_0 = C$ and $\sum_{n=1}^{\infty} |a_n| \leqq |C|/2$, and thus $A_p \leqq |C|$. The identities of Hary-Stein-Spencer with $\lambda = 2$ (see [2], p. 42) and the fact that $p(r, \rho) \geqq 0$ yield the inequality

$$W(r, R) \leqq \int_0^\infty p(r, \rho)\, d\rho^2 = \frac{1}{\pi} \int_0^r \int_{-\pi}^{\pi} |f'(\rho e^{i\theta})|^2 \rho\, d\rho\, d\theta = \sum_{n=1}^{\infty} n\, |a_n|^2 r^{2n}.$$

Since $|f(z)-C| = \left|\sum_{n=1}^{\infty} a_n z^n\right| \leqq \sum_{n=1}^{\infty} |a_n| \leqq |C|/2$, we conclude for $0 \leqq r \leqq 1$ that $n(r, w)=0$ if $|w| < |C|/2$ and hence $W(r, R)=0$ for $0<R<|C|/2$. If $R \geqq |C|/2$, we obtain from the above inequality of $W(r, R)$ and the fact that $f \in M_p^*(C)$:

$$\overline{\lim_{r \to 1}}\, W(r, R) \cdot R^{-2} \leqq 4\,|C|^{-2} \cdot \sum_{n=1}^{\infty} n\,|a_n|^2 \leqq p,$$

and thus the desired inequality $W(R) \leqq p \cdot R^2$. Because of (5) and (8) we obtain from [3], p. 319 the inequality

$$C_{2\varepsilon}(M_p^*(C)) \geqq \frac{\left(ld\,\frac{1}{\varepsilon}\right)^2}{ld\,e\left(\log\frac{1}{r}+\alpha\right)} + 0\left(ld\,\frac{1}{\varepsilon}\,ld\,ld\,\frac{1}{\varepsilon}\right). \tag{9}$$

Now the inequalities $C_{2\varepsilon}(M_p^*(C)) \leqq C_{2\varepsilon}(M_p(C)) \leqq H_\varepsilon(M_p(C))$ yield together with (7) and (9) for $\alpha \to 0$ our next theorem:

THEOREM 2: *Let $H(D)$ be the space of functions holomorphic in the unit disc D. For fixed* r ($0<r<1$) and $p>0$ *we have for the subset $M_p(C)$ of mean p-valent functions in the normed space $(H(D), \|\cdot\|_r)$ the asymptotic formula*

$$H_\varepsilon(M_p(C)) \sim C_\varepsilon(M_p(C)) \sim \frac{\left(ld\,\frac{1}{\varepsilon}\right)^2}{ld\,\frac{1}{r}}.$$

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