TRANSFORMATIONS FOR HYPERGEOMETRIC SERIES (*)

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SOMMARIO - *In questo lavoro si ottengono alcune trasformazioni per le serie ipergeometriche di due variabili . Si crede che i risultati siano nuovi.*

SUMMARY - *In this paper we obtain some transformations for hypergeometric series of two variables. The results are believed to be new.*

1. Introduction.

Bailey [1, 2, 3], Hardy [6], Hill [7], Watson [12], Whipple [13, 14, 15], Suckla [8], Sharma [9], Sharma & Abiodun [10] have obtained various transformations for hypergeometric series. Very few transformations for hypergeometric series for two variables are known. In this paper we have made an attempt to obtain some formulae. These formulae are believed to be new.

The following notation due to Chaundy [5] will be used to represent the hypergeometric series of higher order and of two variables.

$$(1) \qquad F\left[\begin{matrix}(a_p)\,;\,(b_q)\,;\,(c_r)\,;\,x,\,y\\(d_s)\,;\,(e_h)\,;\,(f_l)\,;\end{matrix}\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[(a_p)]_{m+n}\,[(b_q)]_m\,[(c_r)]_n\,x^m\,y^n}{[(d_s)]_{m+n}\,[(e_h)]_m\,[(f_l)]_n\,m\,!\,n\,!},$$

where (a_p) and $[(a_p)]_{m+n}$ will mean $a_1, \dots, a_p$ and $(a_1)_{m+n}, \dots, (a_p)_{m+n}$ respectively.

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2. The first formula to be proved is

$$(2)\quad F\left[\begin{matrix}\alpha\,;-m,b_1,c_1\,;-n,b_2,c_2\,;1,1\\ \beta\,;1-m-b_1,1-m-c_1;1-n-b_2,1-n-c_2\,;\end{matrix}\right]=\frac{(\beta-\alpha)_{m+n}}{(\beta)_{m+n}}\cdot$$

$$\cdot F\left[\begin{matrix}\alpha,1-\beta-m-n\,;-\frac{1}{2}m,\frac{1}{2}-\frac{1}{2}m,1-m-b_1-c_1\,;\\ -\frac{1}{2}n,\frac{1}{2}-\frac{1}{2}n,1-n-b_2-c_2\,;1,1\\ \frac{1}{2}(1-\beta+\alpha-m-n),\frac{1}{2}(2-\beta+\alpha-m-n)\,;\\ 1-m-b_1,1-m-c_1\,;1-n-b_2,1-n-c_2\,;\end{matrix}\right],$$

where $R(\beta-\alpha)>0$, $R(\beta)>0$ and the hypergeometric series on both sides exist.

To prove (2), we start with the left hand side of (2).

$$F\left[\begin{matrix}\alpha\,;-m,b_1,c_1\,;-n,b_2,c_2\,;1,1\\ \beta\,;1-m-b_1,1-m-c_1\,;1-n-b_2,1-n-c_2\,;\end{matrix}\right]=$$

$$=\sum_{u=0}^{m}\sum_{v=0}^{n}\frac{(\alpha)_{u+v}(-m)_u(b_1)_u(c_1)_u(-n)_v(b_2)_v(c_2)_v}{u!\,v!\,(\beta)_{u+v}(1-m-b_1)_u(1-m-c_1)_u(1-n-b_2)_v(1-n-c_2)_v},$$

$$=\sum_{u=0}^{m}\sum_{v=0}^{n}\frac{(\alpha)_{u+v}(-m)_u(-n)_v}{u!\,v!\,(\beta)_{u+v}}\sum_{r=0}^{m-u}\frac{(-u)_r(u-m)_r(1-m-b_1-c_1)r}{(1-m-b_1)_r(1-m-c_1)_r\,r!}\cdot$$

$$\cdot\sum_{s=0}^{n-v}\frac{(-v)_s(v-n)_s(1-n-b_2-c_2)_s}{(1-n-b_2)_s(1-n-c_2)_s\,s!}.$$

We have used the formula, Bailey [4, p. 24 (4.3)]

$$(3)\quad {}_3F_2\left[\begin{matrix}1+a-b-c,a+n,-n\,;1\\ 1+a-b,1+a-c\,;\end{matrix}\right]=\frac{(b)_n(c)_n}{(1+a-b)_n(1+a-c)_n}.$$

$$=\sum_{u=0}^{u+r<m}\sum_{r=0}\sum_{v=0}^{v+s<n}\sum_{s=0}\frac{(\alpha)_{r+s}(1-m-b_1-c_1)_r(1-n-b_2-c_2)_s(-1)^{r+s}(\alpha+r+s)_{u+v}}{(\beta)_{r+s}(1-m-b_1)_r(1-m-c_1)_r(1-n-b_2)_s(1-n-c_2)_s}\cdot$$

$$\cdot\frac{(-m)_{u+r}(-m+u+r)_r(-n)_{v+s}(-n+v+s)_s}{(\beta+r+s)_{u+v}\,r!\,s!\,u!\,v!}$$

we know that

$$\frac{(-u-r)_r}{u+r\,!}=\frac{(-1)^r}{u\,!}\,,\quad \frac{(-v-s)_s}{v+s\,!}=\frac{(-1)^s}{v\,!}\,,$$

$$(-m)_{u+r}(-m+u+r)_r=(-m)_{2r}(-m+2r)_u\,,$$

$$(-n)_{v+s}(-n+v+s)_s=(-n)_{2s}(-n+2s)_v\,,$$

$$(-m)_{2r}=\left(-\frac{1}{2}m\right)_r\left(\frac{1}{2}-\frac{1}{2}m\right)_r 2^{2r}\,,$$

and

$$(-n)_{2s}=\left(-\frac{1}{2}n\right)_s\left(\frac{1}{2}-\frac{1}{2}n\right)_s 2^{2s}\,.$$

$$=\sum_{r=0}^{m/2}\sum_{s=0}^{n/2}\cdot\frac{(\alpha)_{r+s}(1-m-b_1-c_1)_r(1-n-b_2-c_2)_s\left(-\frac{1}{2}m\right)_r\left(\frac{1}{2}-\frac{1}{2}m\right)_r}{(\beta)_{r+s}(1-m-b_1)_r(1-m-c_1)_r(1-n-b_2)_s(1-n-c_2)_s\,r\,!\,s\,!}\cdot$$

$$\cdot\left(-\frac{1}{2}n\right)_s\left(\frac{1}{2}-\frac{1}{2}n\right)_s(-1)^{r+s}\cdot$$

$$\cdot F_1[\alpha+r+s,-m+2r,-n+2s;\beta+r+s;1,1]\,.$$

Now we use the formula

$$F[\alpha;\beta,\gamma;\delta;1,1]=\frac{\Gamma(\delta)\,\Gamma(\delta-\alpha-\beta-\gamma)}{\Gamma(\delta-\alpha)\,\Gamma(\delta-\beta-\gamma)}\,,\tag{4}$$

$$=\frac{(\beta-\alpha)_{m+n}}{(\beta)_{m+n}}\sum_{r=0}^{m/2}\sum_{s=0}^{n/2}$$

$$\frac{(\alpha)_{r+s}(1-m-b_1-c_1)_r\left(-\frac{1}{2}m\right)_r\left(\frac{1}{2}-\frac{1}{2}m\right)_r\left(-\frac{1}{2}n\right)_s}{(1-m-b_1)_r(1-m-c_1)_r(1-n-b_2)_s}\cdot$$

$$\cdot\frac{\left(\frac{1}{2}-\frac{1}{2}n\right)_s(1-n-b_2-c_2)_s(-1)^{r+s}\,2^{2r+2s}}{(1-n-c_2)_s\,r\,!\,s\,!}\cdot$$

$$\cdot\frac{(\beta-\alpha+m+n)_{-2r-2s}\,,}{(\beta+m+n)_{-r-s}}\,,$$

we know that

$$(\beta + m + n)_{-r-s} = \frac{(-1)^{r+s}}{(1-\beta-m-n)_{r+s}},$$

$$(\beta - \alpha + m + n)_{-2r-2s} = \frac{(-1)^{2r+2s}}{(1-\beta+\alpha-m-n)_{2r+2s}},$$

and

$$(1-\beta+\alpha-m-n)_{2r+2s} =$$

$$= \left[\frac{1}{2}(1-\beta+\alpha-m-n)\right]_{r+s} \left[\frac{1}{2}+\frac{1}{2}(1-\beta+\alpha-m-n)\right]_{r+s} 2^{2r+2s}.$$

$$= \frac{(\beta-\alpha)_{m+n}}{(\beta)_{m+n}} F \left[\begin{matrix} \alpha, 1-\beta-m-n; -\frac{1}{2}m, \frac{1}{2}-\frac{1}{2}m, \\ 1-m-b_1-c_1; -\frac{1}{2}n, \frac{1}{2}-\frac{1}{2}n, 1-n-b_2-c_2; 1, 1 \\ \frac{1}{2}(1-\beta+\alpha-m-n), \frac{1}{2}(2-\beta+\alpha-m-n); \\ 1-m-b_1, 1-m-c_1; 1-n-b_2, 1-n-c_2; \end{matrix}\right]$$

by interpreting the final double series with the help of (1). This completes the proof of (2). In particular if we put $m = 0$ or $n = 0$ in (2), it reduces to a known result [4, p. 33 (4.3) (1)].

3. The second formula to be proved is

$$(5)\quad F\left[\begin{matrix}\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2; 1, 1 \\ \delta; \varrho; \mu;\end{matrix}\right] = \frac{\Gamma(\delta)\Gamma(\varrho+\mu+\delta-\alpha-\beta_1-\beta_2-\gamma_1-\gamma_2)}{\Gamma(\delta-\alpha)\Gamma(\varrho+\mu+\delta-\beta_1-\beta_2-\gamma_1-\gamma_2)} \cdot$$

$$\cdot F\left[\begin{matrix}\alpha; \varrho-\beta_1, \varrho-\beta_2; \mu-\gamma_1, \mu-\gamma_2; 1, 1 \\ \varrho+\mu+\delta-\gamma_1-\gamma_2-\beta_1-\beta_2; \varrho; \mu;\end{matrix}\right],$$

where $R(\delta) > 0$, $R(\varrho+\mu+\delta-\alpha-\beta_1-\beta_2-\gamma_1-\gamma_2) > 0$ and the hypergeometric series on both sides exist.

PROOF. To prove (5) we start with the L. H. S. of (5)

$$F\left[\begin{matrix}\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2; 1, 1 \\ \delta; \varrho; \mu\end{matrix}\right] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_m(\gamma_1)_n(\gamma_2)_n}{(\delta)_{m+n}(\varrho)_m(\mu)_n\, m!\, n!},$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta_1)_m(\beta_2)_m}{(\delta)_m(\varrho)_m\, m!}\ {}_3F_2[\gamma_1, \gamma_2, \alpha+m; \mu, \delta+m; 1].$$

Now we make use of the result due to Hardy [4, p. 98, Q. 7].

$$(6)\qquad {}_3F_2\left[\begin{matrix}\alpha_1, \alpha_2, \alpha_3; 1\\ \beta_1, \beta_2;\end{matrix}\right] = \frac{\Gamma(\beta_2)\,\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\beta_2-\alpha_3)\,\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2)}\cdot$$

$$\cdot\, {}_3F_2\left[\begin{matrix}\beta_1-\alpha_1, \beta_1-\alpha_2, \alpha_3; 1\\ \beta_1, \beta_1+\beta_2-\alpha_1-\alpha_2;\end{matrix}\right].$$

$$= \frac{\Gamma(\delta)\,\Gamma(\mu+\delta-\gamma_1-\gamma_2-\alpha)}{\Gamma(\delta-\alpha)} \sum_{m=0}^{\infty} \frac{(\alpha)_m\,(\beta_1)_m\,(\beta_2)_m}{m!\,(\varrho)_m\,\Gamma(\mu+\delta-\gamma_1-\gamma_2+m)}\cdot$$

$$\cdot\, {}_3F_2\,[\mu-\gamma_1, \mu-\gamma_2, \alpha+m; \mu, \mu+\delta-\gamma_1-\gamma_2+m; 1],$$

$$= \frac{\Gamma(\delta)\,\Gamma(\mu+\delta-\gamma_1-\gamma_2-\alpha)}{\Gamma(\delta-\alpha)\,\Gamma(\mu+\delta-\gamma_1-\gamma_2)} \sum_{n=0}^{\infty} \frac{(\mu-\gamma_1)_n\,(\mu-\gamma_2)_n\,(\alpha)_n}{(\mu)_n\,(\mu+\delta-\gamma_1-\gamma_2)_n\,n!}\cdot$$

$$\cdot\, {}_3F_2\,[\beta_1, \beta_2, \alpha+n; \varrho, \mu+\delta-\gamma_1-\gamma_2+n; 1],$$

$$= \frac{\Gamma(\delta)\,\Gamma(\varrho+\mu+\delta-\gamma_1-\gamma_2-\beta_1-\beta_2-\alpha)}{\Gamma(\delta-\alpha)\,\Gamma(\varrho+\mu+\delta-\gamma_1-\gamma_2-\beta_1-\beta_2)} \sum_{n=0}^{\infty} \frac{(\mu-\gamma_1)_n(\mu-\gamma_2)_n(\alpha)_n}{(\mu)_n\,n!}\cdot$$

$$\cdot\, \frac{1}{(\varrho+\mu+\delta-\gamma_1-\gamma_2-\beta_1-\beta_2)_n}\cdot$$

$$\cdot\, {}_3F_2\left[\begin{matrix}\varrho-\beta_1, \varrho-\beta_2, \alpha+n; 1\\ \varrho, \varrho+\mu+\delta-\gamma_1-\gamma_2-\beta_1-\beta_2+n;\end{matrix}\right] \text{ by (6)},$$

$$= \frac{\Gamma(\delta)\,\Gamma(\varrho+\mu+\delta-\gamma_1-\gamma_2-\beta_1-\beta_2-\alpha)}{\Gamma(\delta-\alpha)\,\Gamma(\varrho+\mu+\delta-\gamma_1-\gamma_2-\beta_1-\beta_2)}\cdot$$

$$\cdot\, F\left[\begin{matrix}\alpha; \varrho-\beta_1, \varrho-\beta_2; \mu-\gamma_1, \mu-\gamma_2; 1, 1\\ \varrho+\mu+\delta-\gamma_1-\gamma_2-\beta_1-\beta_2; \varrho; \mu;\end{matrix}\right] \text{ by (1).}$$

This completes the proof of (5). (5) may be called the extension of Hardy's Formula (6) in two variables.

4. The third formula to be proved is

$$(7)\qquad F\left[\begin{matrix}\alpha, \beta; \gamma_1; \gamma_2; 1, 1\\ \delta, \varrho; -; -;\end{matrix}\right] = \frac{\Gamma(\delta)\,\Gamma(\delta+\varrho-\alpha-\beta-\gamma_1-\gamma_2)}{\Gamma(\delta+\varrho-\alpha-\beta)\,\Gamma(\delta-\gamma_1-\gamma_2)}\cdot$$

$$\cdot F\left[\begin{matrix}\varrho-\alpha,\varrho-\beta;\gamma_1;\gamma_2,\varrho+\delta-\alpha-\beta-\gamma_1-\gamma_2, \\ \varrho+\delta-\alpha-\beta-\gamma_1-\gamma_2;1,1 \\ \varrho,\varrho+\delta-\alpha-\beta-\gamma_1-\gamma_2;-;\varrho+\delta-\alpha-\beta-\gamma_2, \\ \varrho+\delta-\alpha-\beta;\end{matrix}\right],$$

where $R(\delta)>0, R(\delta+\varrho-\alpha-\beta-\gamma_1-\gamma_2)>0$ and the hypergeometric series of two variables on both sides exist.

PROOF. To prove (7), we start with the L. H. S. of (7).

$$F\left[\begin{matrix}\alpha,\beta;\gamma_1;\gamma_2;1,1 \\ \delta,\varrho;-;-;\end{matrix}\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_{m+n}(\gamma_1)_m(\gamma_2)_n}{(\delta)_{m+n}(\varrho)_{m+n}\,m!\,n!},$$

$$=\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m(\gamma_1)_m}{(\delta)_m(\varrho)_m\,m!}\;{}_3F_2\left[\begin{matrix}\alpha+m,\beta+m,\gamma_2;1 \\ \delta+m,\varrho+m;\end{matrix}\right],$$

$$=\frac{\Gamma(\delta)\Gamma(\delta+\varrho-\alpha-\beta-\gamma_2)}{\Gamma(\delta-\gamma_2)\Gamma(\delta+\varrho-\alpha-\beta)}\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m(\gamma_1)_m}{(\delta-\gamma_2)_m(\varrho)_m\,m!}\cdot$$

$$\cdot\,{}_3F_2[\varrho-\alpha,\varrho-\beta,\gamma_2;\varrho+m,\delta+\varrho-\alpha-\beta;1]\text{ by (6)},$$

$$=\frac{\Gamma(\delta)\Gamma(\delta+\varrho-\alpha-\beta-\gamma_2)}{\Gamma(\delta-\gamma_2)\Gamma(\delta+\varrho-\alpha-\beta)}\sum_{n=0}^{\infty}\frac{(\varrho-\alpha)_n(\varrho-\beta)_n(\gamma_2)_n}{(\varrho)_n(\delta+\varrho-\alpha-\beta)_n\,n!}$$

$${}_3F_2[\alpha,\beta,\gamma_1;\varrho+n,\delta-\gamma_2;1],$$

$$=\frac{\Gamma(\delta)\Gamma(\varrho+\delta-\gamma_1-\gamma_2-\alpha-\beta)}{\Gamma(\delta-\gamma_1-\gamma_2)\Gamma(\delta+\varrho-\alpha-\beta)}\sum_{n=0}^{\infty}\frac{(\varrho-\alpha)_n(\varrho-\beta)_n(\gamma_2)_n}{(\varrho)_n(\varrho+\delta-\alpha-\beta)_n\,n!}\cdot$$

$$\cdot\frac{(\varrho+\delta-\gamma_1-\gamma_2-\alpha-\beta)_n}{(\varrho+\delta-\gamma_2-\alpha-\beta)_n}\;{}_3F_2\left[\begin{matrix}\varrho-\alpha+n,\varrho-\beta+n,\gamma_1;1 \\ \varrho+n,\varrho+\delta-\gamma_1-\gamma_2-\alpha-\beta+n;\end{matrix}\right]\text{ by (6)},$$

$$=\frac{\Gamma(\delta)\Gamma(\delta+\varrho-\gamma_1-\gamma_2-\alpha-\beta)}{\Gamma(\delta+\varrho-\alpha-\beta)\Gamma(\delta-\gamma_1-\gamma_2)}\cdot$$

$$\cdot F\left[\begin{matrix}\varrho-\alpha,\varrho-\beta;\gamma_1;\gamma_2,\varrho+\delta-\gamma_1-\gamma_2-\alpha-\beta, \\ \varrho+\delta-\gamma_1-\gamma_2-\alpha-\beta;1,1 \\ \varrho,\varrho+\delta-\gamma_1-\gamma_2-\alpha-\beta;-; \\ \delta+\varrho-\alpha-\beta,\varrho+\delta-\gamma_2-\alpha-\beta;\end{matrix}\right]\text{by (1)}.$$

This completes the proof of (7). (7) can be called the extension of Hardy's formula (6) in two variables.

5. The fourth formula to be proved is

$$F\begin{bmatrix}\alpha\,;\,c_1-a_1\,,\,c_1-b_1\,;\,c_2-a_2\,,\,c_2-b_2\,;\,1,1\\ \beta\,;\,c_1\,;\,c_2\end{bmatrix}= \tag{8}$$

$$=\frac{\Gamma(\beta)\,\Gamma(\beta-\alpha-c_1-c_2+a_1+a_2+b_1+b_2)}{\Gamma(\beta-\alpha)\,\Gamma(\beta-c_1-c_2+a_1+a_2+b_1+b_2)}\cdot$$

$$\cdot\,F\begin{bmatrix}\alpha\,;\,a_1\,,\,b_1\,;\,a_2\,,\,b_2\,;\,1,\,1\\ \beta-c_1-c_2+a_1+a_2+b_1+b_2\,;\,c_1\,;\,c_2\,;\end{bmatrix},$$

valid for $R(\beta)>0$, $R(\beta-\alpha-c_1-c_2+a_1+a_2+b_1+b_2)>0$, and the hypergeometric series of two variables on both sides exist.

Proof: We start with the L. H. S. of (8).

$$F\begin{bmatrix}\alpha\,;\,c_1-a_1\,,\,c_1-b_1\,;\,c_2-a_2\,,\,c_2-b_2\,;\,1,1\\ \beta\,;\,c_1\,;\,c_2\,;\end{bmatrix}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(c_1-a_1)_m\,(c_1-b_1)_m\,(c_2-a_2)_n\,(c_2-b_2)_n}{(\beta)_{m+n}\,(c_1)_m\,(c_2)_n\;m\,!\;n\,!}\,.$$

Now we make use of Saalschütz's theorem [4, p. 9 (2.2)]

$$\frac{(c-a)_n\,(c-b)_n}{(c)_n\,n\,!}=\sum_{r=0}^{n}\frac{(a)_r\,(b)_r\,(c-a-b)_{n-r}}{(c)_r\;r\,!\;\overline{n-r}\,!} \tag{9}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\beta)_{m+n}}\sum_{r=0}^{n}\frac{(a_1)_r\,(b_1)_r\,(c_1-a_1-b_1)_{n-r}}{(c_1)_r\;r\,!\;\overline{n-r}\,!}\,.$$

$$\sum_{s=0}^{m}\frac{(a_2)_s(b_2)_s(c_2-a_2-b_2)_{m-s}}{(c_2)_s\;s\,!\;\overline{m-s}\,!}=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha)_{r+s}\,(a_1)_r\,(b_1)_r\,(a_2)_s\,(b_2)_s}{(\beta)_{r+s}(c_1)_r\,(c_2)_s\,r\,!\,s\,!}\,.$$

$$\cdot\,F_1\,[\alpha+r+s\,;\,c_1-a_1-b_1\,,\,c_2-a_2-b_2\,;\,\beta+r+s\,;\,1,\,1],$$

$$=\frac{\Gamma(\beta)\,\Gamma(\beta-\alpha-c_1-c_2+a_1+a_2+b_1+b_2)}{\Gamma(\beta-\alpha)\,\Gamma(\beta-c_1-c_2+a_1+a_2+b_1+b_2)}\cdot$$

$$\cdot\,F\begin{bmatrix}\alpha\,;\,a_1\,,\,b_1\,;\,a_2\,,\,b_2\,;\,1,1\\ \beta-c_1-c_2+a_1+a_2+b_1+b_2\,;\,c_1\,;\,c_2\,;\end{bmatrix}$$

by using the formula (4) and interpreting the double series with the help of (1).

This completes the proof of (8).

6. The fifth formula to be proved is

$$(10) \qquad F\begin{bmatrix}\alpha\,;\, c-a,\, -m\,;\, e-f,\, -n\,;\, 1,1\\ \beta\,;\, c\,;\, e\,;\end{bmatrix} = \frac{(\beta-\alpha)_{m+n}}{(\beta)_{m+n}}\,\cdot$$

$$\cdot\, F\begin{bmatrix}\alpha\,;\, a,\, -m\,;\, f,\, -n\,;\, 1,1\\ 1+\alpha-\beta-m-n\,;\, c\,;\, e\,;\end{bmatrix},$$

where $R(\beta) > 0$, $R(\beta-\alpha-b-g) > 0$ and the hypergeometric series of two variables on both sides exist.

PROOF: We start with the L.H.S. of (10)

$$F\begin{bmatrix}\alpha\,;\, c-a,\, -m\,;\, e-f,\, -n\,;\, 1,1\\ \beta\,;\, c\,;\, e\,;\end{bmatrix},$$

and make use ef Vandermonde's theorem [4, p.3 (1.3)].

$$(11) \qquad \frac{(c-a)_n}{(c)_n} = {}_2F_1(-n,\, a,\, c\,;\, 1).$$

$$= \sum_{u=0}^{m}\sum_{v=0}^{n} \frac{(\alpha)_{u+v}\,(c-a)_u\,(-m)_u\,(e-f)_v\,(-n)_v}{(\beta)_{u+v}\,(c)_u\,(e)_v\,u\,!\,v\,!},$$

$$= \sum_{u=0}^{m}\sum_{v=0}^{n} \frac{(\alpha)_{u+v}\,(-m)_u\,(-n)_v}{(\beta)_{u+v}\,u\,!\,v\,!} \sum_{r=0}^{u} \frac{(a)_r\,(-u)_r}{(c)_r\,r\,!} \sum_{s=0}^{v} \frac{(-v)_s(f)_s}{(e)_s\,s\,!},$$

$$= \sum_{r=0}^{m}\sum_{s=0}^{n} \frac{(\alpha)_{r+s}\,(a_r)\,(f)_s\,(-m)_r(-n)_s\,(-1)^{r+s}}{(c)_r(e)_s\,(\beta)_{r+s}\,r\,!\,s\,!}\,\cdot$$

$$\cdot\, F_1[\alpha+r+s\,;\, -m+r,\, -n+s\,;\, \beta+r+s\,;\, 1,1]$$

$$= \frac{(\beta-\alpha)_{m+n}}{(\beta)_{m+n}}\, F\begin{bmatrix}\alpha\,;\, a,\, -m\,;\, f,\, -n\,;\, 1,\, 1\\ 1+\alpha-\beta-m-n\,;\, c\,;\, e\,;\end{bmatrix}$$

by using the formula (4) and interpreting the double series with the help of (1).

This completes the proof of (10).

In case we take $n=0$ in (10), we get the following interesting transformation of ${}_3F_2$.

$$(13)\qquad {}_3F_2\left[\begin{matrix}\alpha, a, -m\,;1\\ \beta, c\,;\end{matrix}\right]=\frac{(\beta-\alpha)_{m+n}}{(\beta)_{m+n}}\,{}_3F_2\left[\begin{matrix}\alpha, c-a, -m\,;1\\ c, 1+\alpha-\beta-m\end{matrix}\right].$$

REFERENCES

[1] BAILEY, W. N., *Transformations of generalized hypergeometric series.* Proc. London. Math. Soc. (2), 29. pp. 495-502. 1929.

[2] BAILEY, W. N., *Some transformations of generalized hypergeometric series and contour integrals of Barne's type.* Quart. J. Math. (Oxford),3. pp. 168-182. 1932.

[3] BAILEY W. N., *Transformations of well-poised hypergeometric series.* Proc. London. Math. Soc. (2), 36, pp. 235-240. 1934.

[4] BAILEY, W. N. *Generalized hypergeometric series.* Cambridge University Press. 1935.

[5] CHAUNDY, T. W., *Expansion of hypergeometric functions.* Quart. J. Math. (Oxford). 13. 1942.

[6] HARDY, G. H., *A chapter from Ramanujan's note book.* Proc. Camb. Phil. Soc. 21. pp. 492-503. 1923.

[7] HILL, M. J. M. *On a formula for the sum of a finite number of terms of the hypergeometric series when the fourth element is equal to unity.* Proc. London. Math. Soc. (2), 5, pp. 335-341 and 6, pp. 339-348. 1968.

[8] SHUCKLA, H. S., *Certain transformations of nearly-poised bilateral hypergeometric series.* Ganita, 7, pp. 113-121. 1956.

[9] SHARMA, B. L., *Some formulae for hypergeometric series.* Communicated for publication.

[10] SHARMA, B. L. and ABIODUN, R. F. A. *Formulae for hypergeometric functions of two variables.* Communicated for publication.

[11] SLATER, L. J., *Generalized hypergeometric functions.* Cambridge University Press. 1966.

[12] WATSON, G. N.,*General transformations.* Proc. London Math. Soc. (2). 35, pp. 156-199. 1933.

[13] WHIPPLE, F. J. W., *Some transformations of generalized hypergeometric series.* Proc. London. Math. Soc. (2), 26, pp. 257-272. 1927.

[14] WHIPPLE, F. J. W., *Algebraic proofs of the theorems of Cayley and Orr concerning the products of certain hypergeometric series.* J. London. Math. Soc. 2. pp. 85-96. 1927.

[15] WHIPPLE, F. J. W., *On transformations of terminating well-poised hypergeometric series of the type ${}_9F_8$.* J. London. Math. Soc. 9, pp. 137-140. 1934.